Theorem (Existence Theorem)

Every finite dimensional vector space $V(F)$ has a basis.

or

There exists a basis for each FDVS.

Proof Since V is FDVS, there exists a finite subset T of V such that

$$V = L(T)$$

If the elements (vectors) of T are L.I., T becomes a basis of V. This theorem is proved.

Let S is minimal subset of T i.e ($S \subseteq T$) such that $V = L(S)$. If K is proper subset of S. then $L(K) \neq V$

We proceed to show that S is L.I. then

Let $S = \{v_1, v_2, \dots v_n\}$

$$\alpha_1 v_1 + \alpha_2 v_2 + \dots \alpha_n v_n = 0 \quad \text{---} (1)$$

$\alpha_i \in F$ & $v_i \in V$

Let if possible $\alpha_i \neq 0$

then $\alpha_i^{-1} \in F$; $\alpha_i^{-1} \alpha_i = 1$

for some i, $1 \leq i \leq n$.

(*) The proper subset of A is a subset of A that is not equal to A. i.e

*If B is proper subset of A then all elements of B are in A but A contains at least one element that is not in B

Exp sor $A = \{1, 3, 5\} \to$ super

K or $B = \{1, 3\} \to$ minimal

B is proper subset of A

i.e $B \subset A$

from (1)

$$\alpha_1 v_1 + \alpha_2 v_2 + \dots \alpha_i \cdot v_i + \dots \alpha_n v_n = 0$$

$$-\alpha_i \cdot v_i = \alpha_1 v_1 + \alpha_2 v_2 + \dots \alpha_{i-1} v_{i-1} + \alpha_{i+1} v_{i+1} + \dots \alpha_n v_n$$

$$v_i = (-\alpha_i^{-1} \alpha_1) v_1 + (-\alpha_i^{-1} \alpha_2) v_2 + (-\alpha_i^{-1} \alpha_3) v_3 + \dots + (-\alpha_i^{-1} \alpha_{i-1}) v_{i-1} + (-\alpha_i^{-1} \alpha_{i+1}) v_{i+1} + \dots (-\alpha_i^{-1} \alpha_n) v_n \quad \text{---} (11)$$

Let $v \in V$ be arbitrary / there exists

Since $V = L(S)$

We have, Linear combination of S, i.e. LCS:

$$V = \beta_1 v_1 + \beta_2 v_2 + \cdots \beta_i v_i + \cdots \beta_n v_n \quad \text{(II)}$$

From (II) & (III) i.e. v_i putting in Eqn (III)

$$V = \beta_1 v_1 + \beta_2 v_2 + \cdots \beta_i (-\alpha_i^{-1} \alpha_1 v_1 - \alpha_i^{-1} \alpha_2 v_2 + \cdots$$
$$\cdots - \alpha_i^{-1} \alpha_{i-1} v_{i-1} - \alpha_i^{-1} \alpha_{i+1} v_{i+1} - \alpha_i^{-1} \alpha_n v_n) + \cdots$$
$$\cdots \beta_n v_n$$

$$V = (\beta_1 - \beta_i \alpha_i^{-1} \alpha_1) v_1 + (\beta_2 - \beta_i \alpha_i^{-1} \alpha_2) v_2 + \cdots$$
$$+ (\beta_{i-1} - \beta_i \alpha_i^{-1} \alpha_{i-1}) v_{i-1} + \cdots (\beta_n - \beta_i \alpha_i^{-1} \alpha_n) v_n$$

$$V = \gamma_1 v_1 + \gamma_2 v_2 + \cdots \gamma_{i-1} v_{i-1} + \cdots \gamma_n v_n \quad \text{(IV)}$$

where $\gamma_j = \beta_j - \beta_i \alpha_i^{-1} \alpha_j$

$j = 1$ to n

$V = L(K)$

where

$K = \{v_1, v_2 \cdots v_{i-1}, v_{i+1}, \cdots v_n\}$

$K \subset S$ $\therefore$ K is proper subset of S.

The above expression in Eqn (IV) shows that $V = L(K)$ where $K \subset S$

This contradicts the minimality of S.

Hence $\alpha_i = 0 \ \forall i$ and so S is L.I subset of V such that $V = L(S)$. Therefore S is a basis of vector space.

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Theorem (Extension Theorem)

Any finite L.I. subset of FDVS $V(F)$ can be extended to form a basis of V.

or.

If V is FDVS over F and if set $S = \{v_1, v_2, \ldots v_r\}$ is L.I vectors in V, there exists vectors $v_{r+1}, v_{r+2} \ldots v_n$ in V such that $\{v_1, v_2, \ldots v_r, v_{r+1}, \ldots v_n\}$ is basis of V.

Proof:- Let V is FDVS i.e $\dim V = n$, so that n is the maximum number of Linear independent (L.I.) vectors in any subset of V.

$T = \{v_1, v_2 \ldots v_r \ldots v_n\}$

If the set $\{v_1, v_2, \ldots v_r\}$ spans V, then it forms a basis of V and the result is proved (Notice that $r = n$)

Let $S = \{v_1, v_2, \ldots v_r, v_{r+1}, \ldots v_n\}$ be the maximal L.I subset of V. If we prove that $V = L(S)$, then S is a basis of V.

Let $v \in V$ be arbitrary or there exists. Then

$T = \{v_1, v_2, \ldots v_r, v_{r+1}, \ldots v_n, v\}$

which contains $(n+1) > (n)$ vectors is necessarily Linear dependent (L.D.), therefore there exist $\alpha_1, \alpha_2, \ldots \alpha_n, \alpha \in F$ not all zero such that

$\alpha_1 v_1 + \alpha_2 v_2 + \ldots \alpha_n v_n + \alpha v = 0$ — ①

Then, we shall prove that $\alpha \neq 0$

Let if possible, $\alpha = 0$. Then

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \cdots + \alpha_n v_n = 0$$

$$\Rightarrow \alpha_1 = \alpha_2 = 0, \cdots \alpha_n = 0$$

$\because \{v_1, v_2 \cdots v_n\}$ is L.I.

Thus we arrive at a contradiction and so $\alpha \neq 0$

$\because \alpha \in F \Rightarrow \alpha^{-1} \in F$ and $\alpha^{-1}\alpha = 1$

from Eqn (1) we have

$$\alpha v = -\alpha_1 v_1 - \alpha_2 v_2 - \cdots \alpha_n v_n$$

α^{-1} multiply in both side in above expression

$$\alpha^{-1}\alpha v = -\alpha^{-1}\alpha_1 v_1 - \alpha^{-1}\alpha_2 v_2 - \cdots - \alpha^{-1}\alpha_n v_n$$

$$v = -\alpha^{-1}\alpha_1 v_1 - \alpha^{-1}\alpha_2 v_2 - \cdots - \alpha^{-1}\alpha_n v$$

$$v \in L(S) \quad \forall \; v \in V$$

$$V = L(S)$$

Hence, S is a basis of V.

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(1) **Corollary**:- If $\dim V = n$ and $S = \{v_1, v_2, \cdots v_n\}$ is a L.I. subset of V then S is a basis of V.

Proof:- Since S is L.I. subset of V, it can be extended to form a basis of V. Since $\dim V = n$ and S contains n L.I. vectors, S itself forms a basis of V.

(2) **Corollary**:- If $\dim V = n$ and $S = \{v_1, v_2 \cdots v_n\}$ spans V, then S is a basis of V.

Proof: Since $\dim V = n$, so any basis of V has n elements (vectors). Since $V = L(S)$ there exists a subset of S which forms a basis of V.