Exp. n^{th} differential coefficient (derivative) of

(i) $\frac{1}{ax+b}$ & (ii) $\frac{1}{(ax+b)^2}$

Solution (i) Let $y = \frac{1}{ax+b} = (ax+b)^{-1}$

Differentiating y w.r.t x, successively we get

$y_1 = (-1)a(ax+b)^{-2}$

$y_2 = (-1)(-2)a^2(ax+b)^{-3}$

$y_3 = (-1)(-2)(-3)a^3(ax+b)^{-4}$

...

$y_n = (-1)(-2)(-3)\cdots\cdots(-n)a^n(ax+b)^{-(n+1)}$ — ①

and Again differentiating Eq^n ①, we get

$y_{n+1} = (-1)(-2)(-3)\cdots\cdots(-n)(-n-1)a^{n+1}(ax+b)^{-(n+2)}$

Hence, y_{n+1} is of the same form as y_n, given by Eq^n(1). This is true for $n=1,2,3 \cdots$ so on.

Then by mathematical induction, Eq^{n}① true $\forall n$.

from Eq^n(1), we can write y_n as:-

$y_n = (-1)^n[1\cdot2\cdot3\cdots\cdots n]a^n(ax+b)^{-(n+1)}$

i.e $\boxed{y_n = \frac{(-1)^n(n!)a^n}{(ax+b)^{n+1}}}$

Soln for (ii) Let $y = (ax+b)^{-2}$

Differentiating both sides w.r.t x, we get

$y_1 = (-2)a(ax+b)^{-3}$

$y_2 = (-2)(-3)a^2(ax+b)^{-4}$

$y_3 = (-2)(-3)(-4)a^3(ax+b)^{-5}$

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Now

$$y_n = (-2)(-3)(-4)\cdots\cdots(-(n+1))a^n(ax+b)^{-(n+2)} \quad (2)$$

Again diff. Eqn (2) w.r.t x, we get

$$y_{n+1} = (-2)(-3)(-4)\cdots\cdots[-(n+1)][-(n+2)]a^{n+1}(ax+b)^{-(n+3)}$$

Hence, y_{n+1} is of the same form as y_n, given by Eqn (2).

If Eqn (2) is true for $n = 1, 2, 3 \cdots$ s.m.

Then by mathematical induction, Eq (2) is true for every +ve n.

From Eqn (2) we can write y_n

$$y_n = (-1)^n (n+1)!\, a^n \cdot (ax+b)^{-(n+2)}$$

$$\boxed{y_n = \frac{(-1)^n (n+1)!\, a^n}{(ax+b)^{n+2}}}$$

Exp. If $y = \frac{1}{x^2+a^2}$, find y_n.

Solution Given

$$y = \frac{1}{x^2+a^2}$$

$\because x^2+a^2 = (x+ia)(x-ia)$

$$y = \frac{1}{(x+ia)(x-ia)}$$

$$y = \frac{A}{(x+ia)} + \frac{B}{(x-ia)} = \frac{A(x-ia)+B(x+ia)}{(x+ia)(x-ia)} = \frac{1}{(x^2+a^2)}$$

$$A(x-ia)+B(x+ia) = 1$$

putting $x = ia \Rightarrow B = \frac{1}{2ia}$

& putting $x = -ia \Rightarrow A = -\frac{1}{2ia}$

Then

$$y = \frac{-1}{2ia(x+ia)} + \frac{1}{2ia(x-ia)}$$

$$y = \frac{1}{2ia}\left[\frac{1}{x-ia} - \frac{1}{x+ia}\right]$$

We have $y = \frac{1}{ax+b} \Rightarrow y_n = \frac{(-1)^n \lfloor n}{(ax+b)^{n+1}}$

Then

$$y_n = \frac{1}{2ia}\left[\frac{(-1)^n n!}{(x-ia)^{n+1}} - \frac{(-1)^n n!}{(x+ia)^{n+1}}\right]$$

$$y_n = \frac{(-1)^n n!}{2ia}\left[\frac{1}{(x-ia)^{n+1}} - \frac{1}{(x+ia)^{n+1}}\right] \quad \text{--- (1)}$$

Hence, it is the n^{th} derivative of $\frac{1}{x^2+a^2}$