Q. Determine whether or not the following vectors form a basis of R^3 (1,1,2), (1,2,5) and (5,3,4).

Sol. The set S of vectors form a basis of R^3

$S = \{(1,1,2), (1,2,5), (5,3,4)\}$

then, (i) The set S is linear independent

(ii) S spans/generates vector space V.

If $S = \{\underset{v_1}{(1,1,2)}, \underset{v_2}{(1,2,5)}, \underset{v_3}{(5,3,4)}\}$ vectors are L.I. then all scalars α, β, γ are zero

i.e $\alpha = 0 = \beta = 0$ $\gamma = 0$

or $|V| \neq 0$

By Linear combination, let $\alpha, \beta, \gamma \in F$

$$\alpha v_1 + \beta v_2 + \gamma v_3 = 0$$

$$\alpha(1,1,2) + \beta(1,2,5) + \gamma(5,3,4) = 0$$

$$\alpha + \beta + 5\gamma = 0 \quad \text{--- (i)}$$

$$\alpha + 2\beta + 3\gamma = 0 \quad \text{--- (ii)}$$

$$2\alpha + 5\beta + 4\gamma = 0 \quad \text{--- (iii)}$$

from (i) & (ii) $\Rightarrow$ (i) − (ii)

$$-\beta + 2\gamma = 0 \Rightarrow \beta = 2\gamma$$

putting $\beta = 2\gamma$ in (i) & (ii) we get

$\alpha + 2\gamma + 5\gamma = 0 \Rightarrow \alpha + 7\gamma = 0$, $\alpha = -7\gamma$

$\alpha + 2 \times 2\gamma + 3\gamma = 0 \Rightarrow \alpha + 7\gamma = 0$

Let $\gamma = k$ then $\alpha = -7k$, $\beta = 2k$, $\gamma = k$

$$\alpha \neq \beta \neq \gamma \neq 0$$

Therefore given vectors are Linear dependent. they are not L.I.

Other method for checking vectors are L.D or L.I

$$|V| = \begin{bmatrix} 1 & 1 & 5 \\ 1 & 2 & 3 \\ 2 & 5 & 4 \end{bmatrix}$$

$$|V| = 1(8-15) - 1(4-6) + 5(5-4)$$

$$= -7 + 2 + 5 = 0$$

$$|V| = 0$$

Hence given vectors are L.D. or vectors are not L.I. then given vectors set is not basis of R^3

Q.2 Show that the mapping $T: R_2 \to R_3$

or $T: V(R^2) \to V(R^3)$ defined as

$T(x, y) = (x+y, x-y, y)$ is a linear Transformation.

Sol. Given that $T: R_2 \to R_3$

as $T(x,y) = (x+y, x-y, y)$ —①

Then by Linear Tranformation definition as.

(i) $T(x+y) = T(x) + T(y) \quad \because x, y \in V$

(ii) $T(\alpha x) = \alpha T(x) \quad \because \alpha \in F$

Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$

and $\alpha, \beta \in F$

Now for (i) $T(x+y) = T[(x_1, x_2) + (y_1, y_2)]$

$= T[\underbrace{(x_1+y_1)}_{x}, \underbrace{(x_2+y_2)}_{y}]$

$= [(x_1+y_1+x_2+y_2), (x_1+y_1-x_2-y_2), (x_2+y_2)]$

by condition ①

$T(x+y) = [(x_1+x_2+y_1+y_2), (x_1-x_2+y_1-y_2), (x_2+y_2)]$ —②

Now $T(x) + T(y) = T(x_1, x_2) + T(y_1, y_2)$

$= (x_1+x_2, x_1-x_2, x_2) + (y_1+y_2, y_1-y_2, y_2)$

$$T(x) + T(y) = (x_1+x_2, x_1-x_2, x_2) + (y_1+y_2, y_1-y_2, y_2)$$
$$= [(x_1+x_2+y_1+y_2), (x_1-x_2+y_1-y_2), (x_2+y_2)] \quad \text{--- (3)}$$

from (2) & (3)

$$T(x+y) = T(x) + T(y)$$

How $T(\alpha(x)) = T[\alpha(x_1, x_2)]$

$$= T(\alpha x_1, \alpha x_2)$$
$$= ((\alpha x_1 + \alpha x_2), (\alpha x_1 - \alpha x_2), \alpha x_2]$$
$$= \alpha((x_1+x_2), (x_1-x_2), x_2]$$
$$= \alpha T(x_1, x_2)$$

$$T(\alpha x) = \alpha T(x)$$

Hence T is Linear Transformation