Exp. Test for convergence the series

$$\frac{1}{2}+\frac{1}{2\cdot 2^2}+\frac{1}{3\cdot 2^3}+\cdots$$

Sol. We have.

$$a_n = \frac{1}{n\cdot 2^n}$$

$$\Rightarrow \quad a_{n+1} = \frac{1}{(n+1)\cdot 2^{n+1}}$$

$$\because \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n\cdot 2^n}{(n+1)2^{n+1}}$$

$$= \lim_{n\to\infty} \frac{1}{2(1+1/n)}$$

$$= (1/2) < 1$$

Hence $\lambda<1$, by ratio test then given series $\sum \frac{1}{n2^n}$ is convergent.

Que 1. Test for convergence the series

$$\left(\frac{1}{2}\right)^2 + \left(\frac{1}{3}\cdot\frac{2}{5}\right)^2 + \left(\frac{1}{3}\cdot\frac{2}{5}\cdot\frac{3}{7}\right)^2 + \cdots$$

($\because$ Hint $a_n = \left[\frac{1\cdot 2\cdot 3\cdots n}{3\cdot 5\cdot 7\cdots(2n+1)}\right]^2$

$$a_{n+1} = \left[\frac{1\cdot 2\cdot 3\cdots n(n+1)}{3\cdot 5\cdot 7\cdots(2n+1)(2n+3)}\right]^2$$

Examples of Ratio Test

Exp. Test for convergence the series

(i) $\sum_{n=1}^{\infty} \frac{1\cdot 2\cdot 3 \cdots\cdots n}{7\cdot 10 \cdots\cdots (3n+4)}$

(ii) $\sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n+1}$

Sol. (i) We have

$$a_n = \frac{1\cdot 2\cdot 3 \cdots\cdots n}{7\cdot 10 \cdots\cdots (3n+4)}$$

$$a_{n+1} = \frac{1\cdot 2\cdot 3 \cdots\cdots n(n+1)}{7\cdot 10\cdot 13 \cdots\cdots (3n+4)(3n+7)}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\cancel{1\cdot 2\cdot 3 \cdots\cdots n}(n+1)}{\cancel{7\cdot 10\cdot 13 \cdots (3n+4)}(3n+7)} \times \frac{\cancel{7\cdot 10 \cdots (3n+4)}}{\cancel{1\cdot 2 \cdots n}}$$

$$= \lim_{n\to\infty} \frac{n+1}{3n+7}$$

$$= \lim_{n\to\infty} \frac{(1+1/n)}{(3+7/n)} = \frac{1}{3} < 1$$

By Ratio test, the given series

$\sum_{n=1}^{\infty} \frac{1\cdot 2\cdot 3 \cdots\cdots n}{7\cdot 10\cdot 13 \cdots 3n+4}$ converges.

(II) We have
$$a_n = \frac{2^{n-1}}{3^n+1}$$
$$a_{n+1} = \frac{2^n}{3^{n+1}+1}$$
$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2^n(3^n+1)}{2^n(3^{n+1}+1)} \cdot 2$$
$$= \lim_{n\to\infty} \frac{(1+1/3^n)}{(3+1/3^n)} \cdot 2$$
$$= \frac{2(1+0)}{3(1+0)} \qquad \because \lim_{n\to\infty}\left(\frac{1}{3}\right)^n =$$
$$= \frac{2}{3} < 1$$
By Ratio test the given series converges

Q-1- Test for convergence the series
$$\sum_{n=1}^{\infty} \frac{n!}{n^n}$$
Sol. Hint $a_n = \frac{n!}{n^n}$ & $a_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}}$

Applying Ratio test, $\lim_{n\to\infty}(1+1/n)^n = e = 2.7$