07/04/2021 TDC-I Paper-II

Indeterminate form $\left[\frac{\infty}{\infty}\right]$

Let $\phi(a) = \infty$ & $\psi(a) = \infty$

then $\frac{\phi(x)}{\psi(x)}$ is the form $\frac{\infty}{\infty}$ when $x \to a$

It can be written in form $\frac{0}{0}$ as follows

$$\frac{\phi(x)}{\psi(x)} = \frac{1/\psi(x)}{1/\phi(x)}$$

When it is reduced to the form $\frac{0}{0}$ then we can find out the limit by the L'Hospital rule.

The statement of that limit is as follows

If $\lim_{x \to a} \phi(x) = \phi(a) = \infty$ and

$\lim_{x \to a} \psi(x) = \psi(a) = \infty$ then

$$\lim_{x \to a} \frac{\phi(x)}{\psi(x)} = \lim_{x \to a} \frac{\phi'(x)}{\psi'(x)}$$ where $\phi'(x) = \frac{d}{dx}\phi(x)$, $\psi'(x) = \frac{d}{dx}(\psi(x))$

Provided that the limit exists, finite or infinite.

We have $\lim_{x \to a} \frac{\phi(x)}{\psi(x)} = \lim_{x \to a} \frac{1/\psi(x)}{1/\phi(x)}$ [form $\frac{0}{0}$]

$$= \lim_{x \to a} \frac{-\psi'(x)/[\psi(x)]^2}{-\phi'(x)/[\phi(x)]^2} \text{ by L-H R}$$

$$= \lim_{x \to a} \left[\frac{\phi(x)}{\psi(x)}\right]^2 \cdot \frac{\psi'(x)}{\phi'(x)}$$

$$\lim_{x \to a} \frac{\phi(x)}{\psi(x)} = \lim_{x \to a} \frac{\psi'(x)}{\phi'(x)} \cdot \left[\lim_{x \to a} \frac{\phi(x)}{\psi(x)}\right]^2 \quad -(1)$$

Let $\lim_{x \to a} \frac{\phi(x)}{\psi(x)} = k$

Case I if $k \neq 0$ and $k \neq \infty$ then Eqn ①

$$k = \lim_{x \to a} \frac{\psi'(x)}{\phi'(x)} \cdot k^2 \text{ ——— } ①$$

$$\frac{1}{k} = \lim_{x \to a} \frac{\psi'(x)}{\phi'(x)}$$

$$\therefore k = \lim_{x \to a} \frac{\phi'(x)}{\psi'(x)}$$

Case II if $k = 0$, then adding 1 to both sides

$$k+1 = \lim_{x \to a} \frac{\phi(x)}{\psi(x)} + 1$$

$$= \lim_{x \to a} \frac{\phi(x) + \psi(x)}{\psi(x)}$$

$$= \lim_{x \to a} \frac{\phi'(x) + \psi'(x)}{\psi'(x)} \quad \text{by L.H. Rule}$$

$$k+1 = \lim_{x \to a} \frac{\phi'(x)}{\psi'(x)} + 1$$

$$\therefore k = \lim_{x \to a} \frac{\phi'(x)}{\psi'(x)}$$

Case-III if $k = \infty$ then

$$\lim_{x \to a} \frac{1}{[\phi(x)/\psi(x)]} = \lim_{x \to a} \frac{\psi(x)}{\phi(x)}$$

$$= \lim_{x \to a} \frac{\psi'(x)}{\phi'(x)}$$

$$\therefore \lim_{x \to a} \frac{\phi(x)}{\psi(x)} = \lim_{x \to a} \frac{\phi'(x)}{\psi'(x)}$$

Hence it is proved in all the three cases that

$$\therefore \lim_{x \to a} \frac{\phi(x)}{\psi(x)} = \lim_{x \to a} \frac{\phi'(x)}{\psi'(x)}$$

Exp Evaluate $\lim_{x\to\infty} \frac{x^n}{e^x}$

Solution Given that

$$\lim_{x\to\infty} \frac{x^n}{e^x} = \frac{\infty^n}{e^\infty} = \frac{\infty}{\infty}$$

This is of the form $\frac{\infty}{\infty}$

$\therefore$ The given limit $= \lim_{x\to\infty} \frac{n x^{n-1}}{e^x} = \frac{n\cdot\infty^{n-1}}{e^\infty} = \frac{\infty}{\infty}$

Again apply L-Hospital rule.

$$= \lim_{x\to\infty} \frac{n(n-1)x^{n-2}}{e^\infty} = \frac{\infty}{\infty}$$

$$= \lim_{x\to\infty} \frac{n(n-1)(n-2)x^{n-3}}{e^x} = \frac{\infty}{\infty}$$

$\cdot$ — —

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$$= \lim_{x\to\infty} \frac{n(n-1)(n-2)\cdots\cdots 3\cdot2\cdot1}{e^x}$$

$$= \lim_{x\to\infty} \frac{\angle n}{e^x} = \frac{\angle n}{e^\infty} = \frac{\angle n}{\infty} = 0 \quad \because \frac{1}{\infty}=0$$

$$\lim_{x\to\infty} \frac{x^n}{e^x} = \underline{\underline{0}}$$

Exp Evaluate $\lim_{n\to\infty} \frac{\log n}{n}$

Solution Given that $\lim_{n\to\infty} \frac{\log n}{n} = \frac{\log\infty}{\infty} = \frac{\infty}{\infty}$

Applying L-Hospital rule

$$\lim_{n\to\infty} \frac{\log n}{n} = \lim_{n\to\infty} \frac{1/n}{1} = \lim_{n\to\infty} \frac{1}{n} = \frac{1}{\infty}$$

$$= 0$$

$$\therefore \lim_{n\to\infty} \frac{\log n}{n} = \underline{\underline{0}}$$