Comparison Test

Exp. Test for convergence the series

(i) $\sum_{n=1}^{\infty} (\sqrt{n^3+1} - \sqrt{n^3})$

(ii) $\sum_{n=1}^{\infty} (\sqrt{n^4+1} - \sqrt{n^4-1})$

Sol. Given series

(i) $\sum_{i=1}^{\infty} (\sqrt{n^3+1} - \sqrt{n^3})$

We have $a_n = (\sqrt{n^3+1} - \sqrt{n^3})$

$$= \frac{(\sqrt{n^3+1} - \sqrt{n^3})(\sqrt{n^3+1} + \sqrt{n^3})}{(\sqrt{n^3+1} + \sqrt{n^3})}$$

$$a_n = \frac{1}{(\sqrt{n^3+1} + \sqrt{n^3})} \sim \frac{1}{2\sqrt{n^3}}$$

let $b_n = \frac{1}{\sqrt{n^3}} = \frac{1}{n^{3/2}}$, so that $\sum v_n$ converge

Now $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt{n^3}}{(\sqrt{n^3+1} + \sqrt{n^3})}$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{1+(\frac{1}{n})^3} + 1}$$

$$= \frac{1}{\sqrt{1+0} + 1} = \frac{1}{2} \neq 0 \text{ \& finite}$$

So $\sum a_n$ and $\sum b_n$ converge or diverge

Since $\sum b_n = \sum \frac{1}{\sqrt{n^3}} = \sum \left(\frac{1}{n}\right)^{3/2}$ converge

So, the given series converges.

(ii) Given series

let $a_n = \sqrt{n^4+1} - \sqrt{n^4-1}$

$$a_n = \frac{(\sqrt{n^4+1} - \sqrt{n^4-1})(\sqrt{n^4+1} + \sqrt{n^4-1})}{(\sqrt{n^4+1} + \sqrt{n^4-1})}$$

$$a_n = \frac{2}{[(\sqrt{n^4+1}) + \sqrt{n^4-1}]}$$

Let $b_n = \frac{1}{\sqrt{n^4}} = \frac{1}{n^2}$, so that $\sum \frac{1}{n^2}$ converges.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2}{\sqrt{1+(1/n)^4} + \sqrt{1-(1/n)^4}}$$

$$= \frac{2}{\sqrt{1+0} + \sqrt{1-0}} = \frac{2}{2} = 1 \neq 0$$ and finite

So, $\sum a_n$ and $\sum b_n$ converge or diverge together.

Since $\sum b_n$ converges, so $\sum a_n$ converges.

$\sum \frac{1}{n^2}$ Converge, So $\sqrt{n^4+1} - \sqrt{n^4-1}$ converges.

Exp. Test for convergence the series

(i) $\sum \sin\frac{1}{n}$ (ii) $\sum \sin\frac{1}{n^2}$ (iii) $\sum \frac{1}{\sqrt{n}} \tan\frac{1}{n}$

Sol. For (i) $\sum \sin\frac{1}{n}$

Let $a_n = \sin\frac{1}{n}$ & $b_n = \frac{1}{n}$

Now $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sin\frac{1}{n}}{\frac{1}{n}} = 1 \neq 0$ and finite

So, $\sum a_n$ and $\sum b_n$ converge or diverge together

Since $\sum b_n = \sum \frac{1}{n}$ diverges, so $\sum \sin\frac{1}{n}$ diverges.

(ii) $\sum \sin\frac{1}{n^2}$

let $a_n = \sin\frac{1}{n^2}$ and $b_n = \frac{1}{n^2}$

Now $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sin\frac{1}{n^2}}{1/n^2} = 1 \neq 0$

So, $\sum a_n$ & $\sum b_n$ converge or diverge together.

Since $\sum b_n = \sum \frac{1}{n^2}$ convergers, so $\sum \sin\frac{1}{n^2}$ convergers.

(iii) let $a_n = \frac{1}{\sqrt{n}} \tan\frac{1}{n}$ of $\sum \frac{1}{\sqrt{n}} \tan\frac{1}{n}$

and $b_n = \frac{1}{n\sqrt{n}} = \frac{1}{n^{3/2}}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\frac{1}{\sqrt{n}} \tan\frac{1}{n}}{1/n\sqrt{n}} = \lim_{n\to\infty} \frac{\tan\frac{1}{n}}{\frac{1}{n}} = 1 \neq 0 \text{ \& finite}$$

So, $\sum a_n$ and $\sum b_n$ converge or diverge together.

Since $\sum b_n = \sum \frac{1}{n^{3/2}}$ converges so. Converges.

Q-1. Test for convergence the series

$$\frac{\sqrt{2}-1}{3^3-1} + \frac{\sqrt{3}-1}{4^3-1} + \frac{\sqrt{4}-1}{5^3-1} + \cdots\cdots$$

(∵ Hint $a_n = \frac{\sqrt{n+1}-1}{(n+2)^3-1}$ & $b_n = \frac{\sqrt{n}}{n^3} = \frac{1}{n^{5/2}}$

& find $\lim_{n\to\infty} \frac{a_n}{b_n} = ?$

Q-2. Show that the series

$$\frac{1\cdot 2}{3^2\cdot 4^2} + \frac{3\cdot 4}{5^2\cdot 6^2} + \frac{5\cdot 6}{7^2\cdot 8^2} + \cdots\cdots \text{ Converges.}$$

Sol.. Hint:- $a_n = \frac{(2n-1)\cdot 2n}{(2n+1)^2\cdot(2n+2)^2}$ & $b_n = \frac{1}{n^2}$

& find $\lim_{n\to\infty} \frac{a_n}{b_n} = ?$