Exp. Solve $x\frac{dy}{dx}+y=y^2\log x$

Solution - We have

$$x\frac{dy}{dx}+y=y^2\log x$$

$$\frac{1}{y^2}\frac{dy}{dx}+\frac{1}{y}\cdot\frac{1}{x}=\frac{1}{x}\log x \quad \text{------ (i)}$$

put $\frac{1}{y}=z \therefore -\frac{1}{y^2}\frac{dy}{dx}=\frac{dz}{dx}$

Then the eqn (i) becomes

$$-\frac{dz}{dx}+\frac{z}{x}=\frac{1}{x}\cdot\log x$$

or $\frac{dz}{dx}-\frac{z}{x}=-\frac{1}{x}\log x$ ------ (ii)

Which is Linear diff. Eqn. in z. $P=-\frac{1}{x}, Q=\frac{-1}{x}\log x$

$$I.F=e^{\int P dx}=e^{-\int\frac{1}{x}dx}=e^{-\log x}\Rightarrow e^{\log\frac{1}{x}}=\frac{1}{x}$$

Multiplying (ii) by I.F and integrating, we get

$$z\cdot\frac{1}{x}=-\int\frac{1}{x}\log x\cdot\frac{1}{x}dx$$

$$z\cdot\frac{1}{x}=-\int\log x\cdot\frac{1}{x^2}dx$$

$$=-\left[\log x\cdot\int\frac{1}{x^2}dx-\int\left[\frac{d}{dx}(\log x)\cdot\int\frac{1}{x^2}dx\right]dx\right]$$

$$=-\left[-\frac{1}{x}\log x-\int\frac{1}{x}\cdot\left(-\frac{1}{x}\right)dx\right]$$

$$=\frac{1}{x}\log x-\int\frac{1}{x^2}dx$$

$$z\cdot\frac{1}{x}=\frac{1}{x}\log x+\frac{1}{x}+C$$

Hence $\frac{1}{xy}=\frac{1}{x}[\log x+1]+C$