Exp. Give $dy/dx = f(x,y) = 1+y^2$, where $y=0$ when $x=0$, find $y(0.2)$, $y(0.4)$ and $y(0.6)$ using Runge-kutta fourth-order formula.

Solution. Given that

$$f(x,y) = \frac{dy}{dx} = 1+y^2$$

$$y(0) = 0 \text{ when } x=0$$

we take $h = 0.2$, with $x_0 = y_0 = 0$

The fourth-order Runge-kutta formula

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \quad \text{—— (1)}$$

where

$$k_1 = hf(x_0, y_0) \quad \text{—— (2)}$$

$$k_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{1}{2}k_1) \quad \text{—— (3)}$$

$$k_3 = hf(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2) \quad \text{—— (4)}$$

$$k_4 = hf(x_0 + h, y_0 + k_3) \quad \text{—— (5)}$$

From (2) $k_1 = 0.2[1 + y_0^2] = 0.2[1+0] = 0.2$

From (3) $k_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2})$

$$x_0 + \frac{h}{2} = 0 + \frac{.2}{2} = 0.1$$

$$y_0 + \frac{k_1}{2} = 0 + \frac{.2}{2} = 0.1$$

$$f(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}) = 1 + (.1)^2 = 1.01$$

$$k_2 = 0.2(1.01) = 0.202$$

Similarly from (4) $k_3 = 0.2\left[1 + (0 + \frac{.202}{2})^2\right]$

$$= 0.2\left[1 + (.101)^2\right] = 0.2[1 + 0.010201]$$

$$k_3 = 0.20204$$

From (5) $k_4 = 0.2[1 + (0.20204)^2] =$

$$0.2[1 + 0.040820] = 0.20816$$

Then from (1)

$y_1 = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

$= 0 + \frac{1}{6}[0.2 + 2\times 0.202 + 2\times 0.20204 + 0.20816]$

$y_1 = \frac{1}{6}[0.2 + 0.404 + .40408 + 0.20816]$

$= \frac{1}{6}[1.21624] = 0.2027066$

$y_1 = 0.2027$ which is correct to four decimal places

$y(0.2) = 0.2027$

To compute $y(0.4)$, we take $x_0 = 0.2$ $y_0 = 0.2027$ and $h = 0.2$ with these value for Equations (1) to (5) give

$k_1 = 0.2[1 + (0.2027)^2] = 0.2082$

$k_2 = 0.2[1 + (0.3068)^2] = 0.2188$

$k_3 = 0.2[1 + (0.3121)^2] = 0.2195$

$k_4 = 0.2[1 + (0.4222)^2] = 0.2356$

and

$y(0.4) = 0.2027 + \frac{1}{6}[0.2082 + 2\times 0.2188 + 2\times 0.2195 + 0.2356]$

$= 0.2027 + \frac{1}{6}[0.2082 + .4376 + .4660 + 0.2356]$

$= 0.2027 + \frac{1}{6}[1.3204]$

$= 0.2027 + 0.2200666$

$= .4227666$

$y(0.4) = 0.4228$ Correct to four decimal places

Finally, taking $x_0 = 0.4$, $y_0 = 0.4228$ and $h = 0.2$ and proceeding as previous. From Eqn (1) to (6) give.

$$k_1 = 0.2\left[1 + (0.4228)^2\right] = 0.2356$$

$$k_2 = 0.2\left[1 + \left(.4228 + \frac{.2356}{2}\right)^2\right] = 0.2584$$

$$k_3 = 0.2\left[1 + \left(.4228 + \frac{.2584}{2}\right)^2\right] = .2576$$

$$k_4 = 0.2\left[1 + (.4228 + .2576)^2\right] = .2926$$

then using these values in Eqn (1)

$$y(0.6) = 0.4228 + \frac{1}{6}\left[0.2356 + 2 \times 0.2584 + 2 \times 0.2576 + 0.2926\right]$$

$$= 0.4228 + \frac{1}{6}\left[0.2356 + 0.5168 + 0.5152 + 0.2926\right]$$

$$= 0.4228 + \frac{1}{6}\left[1.5602\right]$$

$$= 0.4228 + 0.260033$$

$$0.682833$$

$$y(0.6) = 0.6828$$