* Linear non-homogeneous differential Equation with constants coefficients:-

Exp. Solve $y'' + 3y' - 4y = e^{-4x} + x e^{-x}$

Sol. Given non-homogeneous linear second-order differential equation-

$$y'' + 3y' - 4y = e^{-4x} + x e^{-x} \quad \text{— (i)}$$

Thus, homogeneous diff. Eqn.

$$y'' + 3y' - 4y = 0$$

$$f(x) = e^{-4x} + x e^{-x}$$

The auxiliary Eqn is

$$m^2 + 3m - 4 = 0$$

$$m = \frac{(-3) \pm \sqrt{9 + 4 \times 4 \times 1}}{2} = \frac{-3 \pm 5}{2}$$

the roots $m_1 = 1, -4$ then complementary function

Hence $y_0 = c_1 e^{x} + c_2 e^{-4x}$ — (ii)

Since, first part (term) of $f(x)$ is e^{-4x}, which is solution of homogeneous equation, we assume the particular solution is

$$y_p = x^s e^{-4x}(A_1) + e^{-x}(A_2 x + A_3)$$

where $s = 1$-fold root Then

$$y_p = x \cdot e^{-4x} A_1 + x e^{-x} A_2 + e^{-x} A_3 \quad \text{— (iii)}$$

$$y_p = A_1 x e^{-4x} + A_2 x e^{-x} + A_3 e^{-x} \quad \text{—— (3)}$$

where A_1, A_2, A_3 are constants to be determined.

Differentiating Eq (3) w.r.to x

$$y_p' = A_1(e^{-4x} \cdot 1 + x \cdot e^{-4x} \cdot (-4)) + A_2\{x \cdot e^{-x} \cdot (-1) + e^{-x}\} + A_3 e^{-x}(-1)$$

$$y_p' = A_1 e^{-4x} - 4A_1 x e^{-4x} + A_2 e^{-x} - A_2 x e^{-x} - A_3 e^{-x}$$

Again differentiating its.

$$y_p'' = A_1 e^{-4x}(-4) - 4A_1(-4x e^{-4x} + e^{-4x}) - A_2 e^{-x} - A_2(-x e^{-x} + e^{-x}) + A_3 e^{-x}$$

$$y_p'' = -4A_1 e^{-4x} + 16 A_1 x e^{-4x} - 4A_1 e^{-4x} - A_2 e^{-x} + A_2 x e^{-x} - A_2 e^{-x} + A_3 e^{-x}$$

on substituting y'', y' and y in Eqn (1) then equating terms of coefficients.

$$-8A_1 e^{-4x} + 16A_1 x e^{-4x} - 2A_2 e^{-x} + A_2 x e^{-x} + A_3 e^{-x} + 3(A_1 e^{-4x} - 4A_1 x e^{-4x} + A_2 e^{-x} - A_2 x e^{-x} - A_3 e^{-x}) - 4(A_1 x e^{-4x} + A_2 x e^{-x} + A_3 e^{-x}) = e^{-4x} + x \cdot e^{-x}$$

$$e^{-4x}(-8A_1 + 3A_1) + x e^{-4x}(16A_1 - 12A_1 - 4A_1) + x e^{-x}(A_2 - 3A_2 - 4A_2) + e^{-x}(-2A_2 + A_3 + 3A_2 - 3A_3 - 4A_3) = e^{-4x} + x \cdot e^{-x}$$

$-5A_1 = 1 \Rightarrow A_1 = -1/5$

$-6A_2 = 1 \Rightarrow A_2 = -1/6$

$A_2 - 6A_3 = 0$

$A_2 = 6A_3$

$A_3 = \frac{A_2}{6} = -\frac{1}{6} \times \frac{1}{6} = -\frac{1}{36}$

Putting these values A_1, A_2, A_3 in Eqn (3)

$$y_p = -\frac{1}{5} x e^{-4x} - \frac{1}{6} x e^{-x} - \frac{1}{36} e^{-x}$$

$$= \frac{-x}{5} e^{-4x} - \frac{1}{6} e^{-x}\left(x + \frac{1}{6}\right)$$

Hence the complete solution is

$$y = y_0 + y_p$$

$$y = c_1 e^{x} + c_2 e^{-4x} - \frac{x}{5} e^{-4x} - \frac{1}{6} e^{-x}\left(x + \frac{1}{6}\right)$$