Non-homogeneous linear diff. Eqn. with const. Coeff.

(iii) As a third case, if $f(x)$ is of the form

$$f(x) = e^{\alpha x}\left(b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0\right)\begin{cases}\sin\beta x \\ \cos\beta x\end{cases}$$

then a suitable form for $y_p(x)$, provided that $\alpha + i\beta$ is not a root of the auxiliary equation, is

$$y_p(x) = e^{\alpha x}\left(A_m x^m + A_{m-1}x^{m-1} + \cdots + A_0\right)\sin\beta x + e^{\alpha x}\left(B_m x^m + B_{m-1}x^{m-1} + \cdots B_0\right)\cos\beta x \quad \text{---} (1)$$

If $\alpha + i\beta$ is an s-fold root of the auxiliary equation it is necessary to multiply the right-hand side of equation (1) by x^s.

$$y_p(x) = x^s e^{\alpha x}\left\{\left(A_m x^m + A_{m-1}x^{m-1} + \cdots A_0\right)\sin\beta x + \left(B_m x^m + B_{m-1}x^{m-1} + \cdots B_0\right)\cos\beta x\right\}$$

Exp. Solve $y'' + 4y = \cos x$

Sol. Given that $y'' + 4y = \cos x$ — (1)

Homogeneous Equation is

$$y'' + 4y = 0$$

and $f(x) = \cos x$

The auxiliary Equation is

$$m^2 + 4 = 0$$

$$m = \pm 2i \qquad m_1 = 0 + 2i, \quad m_2 = 0 - 2i$$

Thus $y_0 = [c_1 \cos 2x + c_2 \sin 2x]$ —— (ii)

from $f(x) = \cos x$

we can write and compare from this.

$$f(x) = e^{\alpha x}[P_m(x)[\cos \beta x] + Q_m(x) \sin \beta x]$$

Here $\alpha = 0, \beta = 1$ so $\alpha \pm i\beta = 0 \pm i$. Since $\alpha \pm i\beta (\cancel{\pm} \pm i)$ are not solutions of the auxiliary Equation. the we can write a particular solution of the form

$$y_p = e^{\alpha x}[A \cos \beta x + B \sin \beta x]$$

$$y_p = A \cos x + B \sin x \quad \text{— (3)}$$

on differentiating Eqn (3) w.r.t. to x. then we get

$$y_p' = -A \sin x + B \cos x$$

$$y_p'' = -A \cos x - B \sin x$$

On substituting of y_p & y_p'' in Eqn (1) and equating coefficients from both side,

$$-A \cos x - B \sin x + 4 A \cos x + 4 B \sin x = \cos x$$

$$3A \cos x + 3B \sin x = \cos x$$

$3A = 1$ $\quad A = 1/3$

$3B = 0$ $\quad B = 0$

Then particular solution is

$$y_p = \frac{1}{3}\cos x$$

and complete solution is

$$y = c_1\cos 2x + c_2\sin 2x + \frac{1}{3}\cos x$$

<u>Exp.</u> The same example for the equation

$$y'' + 4y = \cos 2x$$

<u>Solution</u> The auxiliary equation is same as above Exp.

then $y_0 = c_1\cos 2x + c_2\sin 2x$ —— (1)

Hence $\lambda = \pm 2i$, $\alpha = 0$ and $\beta = 2$ then

$\alpha \pm i\beta = 0 \pm 2i$ are solutions of the auxiliary equation then desired particular solution is

$$y_p = e^{\alpha x} x^s (A\cos\beta x + B\sin\beta x)$$

Thus $\rightarrow$ $y_p = x(A\cos 2x + B\sin 2x)$ —— (2)

$\therefore$ $y_p' = (A\cos 2x + B\sin 2x)\cdot 1 + x(-A\sin 2x\cdot 2 + 2B\cos 2x)$

$y_p'' = -A\sin 2x\cdot 2 + B\cos 2x\cdot 2 - 2(A\sin 2x - B\cos 2x)\cdot 1$
$+ x[-4A\cos 2x - 4B\sin 2x]$

on substituting y_p & y_p'' in Eqn (2) and equating like terms of coefficients.

$$-4A\sin 2x + 4B\cos 2x - (4A\cos 2x + 4B\sin 2x)x + 4x(A\cos 2x + B\sin 2x)$$

$$= \cos 2x$$

$$-4A\sin 2x + 4B\cos 2x - \cancel{4Ax\cos 2x} - \cancel{4Bx\sin 2x} + \cancel{4xA\cos 2x}$$

$$+ \cancel{4x\sin 2x} = \cos 2x$$

$$4B = 1 \Rightarrow B = 1/4$$

$$-4A = 0 \Rightarrow A = 0$$

Thus, a particular solution is

$$y_p = x \cdot \frac{1}{4}\sin 2x$$

Then

$$y = y_c + y_p$$

$$y = c_1\cos 2x + c_2\sin 2x + \frac{x}{4}\sin 2x$$

is complete solution.