Que. Show that the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ $(p>0)$ is convergent if $p>1$ and divergent if $p \le 1$

<u>Solution</u> -: Using Chauchy's Integral Test.

Let $a_n = \frac{1}{n^p}$ and $a(x) = \frac{1}{x^p}$,

so that $a(n) = a_n \quad \forall\, n \in N$

Since, for $x \ge 1$, $a(x)$ is non-negative, integrable and a decreasing function of x.

Now.
$$\lim_{t\to\infty} \int_1^t a(x)\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^p}\,dx$$

$$\because \int_1^t x^{-p}\,dx = \frac{1}{(-p+1)}\left[x^{-p+1}\right]_1^t, \text{ if } p \neq$$

$$\int_1^t \frac{dx}{x} = |\log x|_1^t \quad \text{if } p=1$$

$$\therefore \int_1^t a(x)\,dx = \int_1^t x^{-p}\,dx = \frac{1}{1-p}\left[t^{1-p} - 1\right], \text{ if}$$

$$= (\log t - \log 1), \; p=1$$

Now, $\lim\limits_{t\to\infty}\int_1^t a(x)\,dx = \begin{cases} \frac{1}{1-p}\left[\lim\limits_{t\to\infty}(t^{1-p}-1)\right] & \text{if } p\neq 1 \\ \lim\limits_{t\to\infty}\log t & \text{if } p=1 \end{cases}$

$\because \log 1 = 0$

$$= \begin{cases} \frac{1}{1-p}\left(\infty^{(1-p)}-1\right)=\infty & \text{if } p<1 \\ \frac{1}{1-p}(0-1) = \frac{1}{p-1} & \text{if } p>1 \\ \log\infty = \infty & \text{if } p=1 \end{cases}$$

So, hence

$$\lim_{t\to\infty}\int_1^t a(x)\,dx = \infty \quad \text{if } p\leq 1$$

and

$$\lim_{t\to\infty}\int_1^t a(x)\,dx = \frac{1}{p-1} \quad \text{if } p>1$$

It follows that the improper integral $\int_1^\infty a(x)\,dx$ is convergent if $p>1$ and divergent if $p\leq 1$.

Hence, by Cauchy's integral test. $\sum_{n=1}^{\infty}\frac{1}{n^p}$ is convergent if $p>1$ and divergent if $p\leq 1$.