Example. Apply Cauchy's Integral Te
to examine the convergence of t
following series-

(i) $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ (ii) $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$

Solution:-

Let $a(x) = \frac{1}{x^2+1}$, So that

$a(n) = a_n \quad \forall\, n \in N$

For $x \geq 1$, $a(x)$ is non-negative, monotoni cally decreasing and integrable function
Now

$$\lim_{t\to\infty} \int_1^t \frac{dx}{x^2+1}$$

$$\lim_{t\to\infty} \left[\tan^{-1}x\right]_1^t$$

$$\lim_{n\to\infty} \left[\tan^{-1}t - \tan^{-1}1\right]$$

$$\tan^{-1}\infty - \tan^{-1}1$$

$$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$, which is finit

Thus $\int_{n=1}^{\infty} \frac{dx}{x^2+1}$ is convergent and s

$\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ is convergent.

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(ii) Let $a(x) = \frac{1}{x^2+x}$, so that $a(n) = a_n, \forall n \in N$

for $x \geq 1$, $a(x)$ is non-negative, monotonically decreasing and integrable function. Now

$$\lim_{t\to\infty} \int_1^t \frac{dx}{x^2+x} = \lim_{t\to\infty} \int_1^t \frac{dx}{x(x+1)}$$

$$= \lim_{t\to\infty} \left[\int_1^t \left(\frac{1}{x} - \frac{1}{x+1} \right) dx \right]$$

$$= \lim_{t\to\infty} \left[\left| \log x - \log(x+1) \right|_1^t \right]$$

$$= \lim_{t\to\infty} \left[\left| \log \frac{x}{x+1} \right|_1^t \right]$$

$$= \lim_{t\to\infty} \left[\log \frac{t}{t+1} \right] - \lim_{t\to\infty} \left[\log \frac{1}{2} \right]$$

$$= \lim_{t\to\infty} \left[\log \left(\frac{1}{1+1/t} \right) \right] - \log \frac{1}{2}$$

$$= \log 1 - \log \frac{1}{2} = \log 1 - \log 1 + \log 2$$

$$= \log \frac{1}{1} + \log 2 \quad \because \log 1 = 0$$

$$= \log 2, \text{ which is finite}$$

Thus $\int_{n=1}^{\infty} \frac{dx}{x^2+x}$ is convergent and so $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$ is convergent by Cauchy Integral Test.