Q. Let S and T be linear operators on a finite dimensional inner product space $V(R)$ and $\alpha \in R$ Then

(i) $(T+S)^* = T^* + S^*$

(ii) $(\alpha T)^* = \bar{\alpha}(T^*)$

(iii) $(TS)^* = S^* T^*$

(iv) $(T^*)^* = T$

Proof Let $u, v \in V$ be arbitrary

(i) $\langle (T+S)(u),(v)\rangle = \langle T(u)+S(u),(v)\rangle$

$= \langle T(u),(v)\rangle + \langle S(u),(v)\rangle$ — by inner prod prop

$= \langle u, T^*(v)\rangle + \langle u, S^*(v)\rangle$

$= \langle u, (T^*+S^*)(v)\rangle$

Thus: $\langle (T+S)(u),(v)\rangle = \langle u, (T^*+S^*)(v)\rangle$

But $\langle (T+S)(u), v\rangle = \langle u, (T+S)^*(v)\rangle$

by Adjoint operator

Hence

$\langle u, (T+S)^*(v)\rangle = \langle u, (T^*+S^*)(v)\rangle$

Uniqueness of adjoint implies

$$(T+S)^* = \underline{\underline{T^*+S^*}}$$

(ii) $\langle (\alpha T)(u), v\rangle = \langle \alpha T(u), v\rangle$

$= \alpha \langle T(u), v\rangle$

$= \alpha \langle u, T^*(v)\rangle$

$= \langle u, \bar{\alpha} T^*(v)\rangle$

$= \langle u, (\bar{\alpha} T^*) v\rangle$

$\therefore \langle (\alpha T)(u), v\rangle = \langle u, (\bar{\alpha} T^*) v\rangle$

But $\langle (\alpha T)(u), v\rangle = \langle u, (\alpha T)^* v\rangle$ by Adjoint

$\therefore \langle u, (\bar{\alpha} T^*) v\rangle = \langle u, (\alpha T)^* v\rangle$

$\Rightarrow \bar{\alpha} T^* = (\alpha T)^*$ Uniqueness of Adjoint

(iii) $\langle (TS)(u), v\rangle = \langle T(S)(u), v\rangle$

$= \langle (S)(u), T^*(v)\rangle$

$= \langle (u), S^* T^*(v)\rangle$

Also $\langle (TS)(u), v\rangle = \langle u, (TS)^* v\rangle$

$\therefore \langle u, (TS)^* v\rangle = \langle u, S^* T^* v\rangle$

$\Rightarrow (TS)^* = S^* T^*$

(iv) $\because \langle u, T(v) \rangle = \langle T^*(u), v \rangle$ by Adjoint oper

$= \langle u, (T^*)^* v \rangle$

$\langle u, T(v) \rangle = \langle u, (T^*)^*(v) \rangle$

$T = (T^*)^*$