Exp. Solve $(D^2+16)y = \tan 4x$

Sol. Given differential equation is non-homogeneous.
So general solution is $y = C.F + P.I$

To find C.F from $(D^2+16)y=0$

The auxiliary equation is

$$D^2+16=0$$
$$D^2=-16$$
$$D^2 = 16i^2$$
$$D = \pm 4i \Rightarrow \alpha \pm i\beta = 0 \pm 4i \quad \because \alpha = 0$$

Thus $C.F = (c_1 \cos 4x + c_2 \sin 4x) \cdot e^{0 \cdot x}$

$$C.F = c_1 \cos 4x + c_2 \sin 4x$$

And $P.I = \frac{1}{D^2+16} \tan 4x$

$$= \frac{1}{8i}\left[\frac{1}{D-4i} - \frac{1}{D+4i}\right] \tan 4x$$

$$= \frac{1}{8i}\left[\frac{1}{D-4i}\tan 4x - \frac{1}{D+4i}\tan 4x\right]$$

$$= \frac{1}{8i}\left[e^{4ix}\int e^{-4ix}\tan 4x\,dx - e^{-4ix}\int e^{4ix}\tan 4x\,dx\right]$$

$e^{iax} = \cos ax + i \sin ax$ & $\quad \because \frac{1}{D-a}X = e^{ax}\int e^{-ax}X\,dx$

$$= \frac{1}{8i}\left[e^{4ix}\int[\cos 4x - i\sin 4x]\times\tan 4x\,dx - e^{-4ix}\int[\cos 4x + i\sin 4x]\times\tan 4x\,dx\right]$$

$$= \frac{1}{8i}\left[e^{4ix}\int(\cos 4x.\tan 4x\,dx - i\sin 4x.\tan 4x)\,dx - e^{-4ix}\int(\cos 4x\tan 4x + i\sin 4x.\tan 4x)\,dx\right]$$

$$= \frac{1}{8i}\left\{ e^{4ix}\int\left[\sin 4x - i\sin^2 4x \cdot \frac{1}{\cos 4x}\right]dx - e^{-4ix}\int\left[\sin 4x + i\,\frac{\sin^2 4x}{\cos 4x}\right]dx \right\}$$

$$= \frac{1}{8i}\left\{ e^{4ix}\left(\int \sin 4x\,dx - i\int \sin^2 4x \cdot \frac{1}{\cos 4x}\,dx\right) - e^{-4ix}\left(\int \sin 4x\,dx + i\int \sin^2 4x \cdot \frac{1}{\cos 4x}\,dx\right)\right\}$$

$$= \frac{1}{8i}\left\{ e^{4ix}\left[\int \sin 4x\,dx - i\left(\int \sec 4x\,dx - \int \cos 4x\,dx\right)\right] - e^{-4ix}\left[\int \sin 4x\,dx + i\left(\int \sec 4x\,dx - \int \cos 4x\,dx\right)\right]\right\}$$

$$= \frac{1}{8i}\left\{ e^{4ix}\left[-\frac{\cos 4x}{4} - \frac{i}{4}\left(\log(\sec 4x + \tan 4x) - \sin 4x\right)\right] - e^{-4ix}\left[-\frac{\cos 4x}{4} + \frac{i}{4}\left(\log(\sec 4x + \tan 4x) - \sin 4x\right)\right]\right\}$$

$$= \frac{1}{32i}\left\{ \cos 4x\left(e^{-4ix} - e^{4ix}\right) - i\log(\sec 4x + \tan 4x)\left\{e^{4ix} + e^{-4ix}\right\} + i\sin 4x\left(e^{4ix} + e^{-4ix}\right)\right\}$$

$$= \frac{1}{32i}\left\{ \cos 4x \cdot (-2i\sin 4x) - i\log(\sec 4x + \tan 4x) \cdot 2\cos 4x + i\sin 4x \cdot 2\cos 4x\right\}$$

$$= \frac{1}{32i}\left\{ -i\,\cancel{\sin 8x} - 2i\cos 4x \cdot \log(\sec 4x + \tan 4x) + i\,\cancel{\sin 8x}\right\}$$

$$I = -\frac{1}{16}\cos 4x \cdot \log(\sec 4x + \tan 4x)$$

The general solution is $y = C.F + P.I$

Thus $y = c_1\cos 4x + c_2 \sin 4x - \frac{1}{16}\cos 4x \cdot \log(\sec 4x + \tan 4x)$

Exp. Solve $(D^2+1)y = \cos x$

Solution The g.s. of given non-homogeneous differentia Equation is $y = C.F + P.I$

To find C.F of homogeneous $(D^2+1)y=0$

$$D = \pm i$$

then $C.F = c_1\cos x + c_2 \sin x$

and $P.I = \frac{1}{D^2+1}\cos x.$

$\because \frac{1}{f(D^2)}\cos ax = \frac{1}{f(-a^2)}\cos ax$ if $f(-a^2) \neq 0$

Put $D^2 = -a^2$

Here $f(-a^2) = 0$ then

$$\frac{1}{f(D^2)}\cos ax = x \cdot \frac{1}{f'(-a^2)}\cos ax \quad \text{if } f'(-a^2) \neq 0$$

Then $\frac{1}{D^2+1}\cos x = x \cdot \frac{1}{2D}\cos x = \frac{x}{2}\cdot D^{-1}\cos x$

$$= \frac{x}{2}\int \cos x\,dx \qquad D^{-1} = \int dx$$

$$D = \frac{dy}{dx}$$

P.I $= \frac{x}{2}\cdot \sin x$

Thus general solution is $y = C.F + P.I$

$$y = c_1\cos x + c_2\sin x + \frac{x}{2}\sin x.$$