Examples of Homogeneous diff. Eqn.

Exp. Solve $\frac{dy}{dx} = \frac{y}{x} + \tan\frac{y}{x}$

Solution. Given that

$$\frac{dy}{dx} = \frac{y}{x} + \tan\frac{y}{x} \quad \text{--- (i)}$$

Let us put $\frac{y}{x} = v$

$$y = vx$$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$\therefore$ From (i)

$$v + x\frac{dv}{dx} = v + \tan v$$

$$x\frac{dv}{dx} = \tan v$$

$$\frac{dv}{\tan v} = \frac{dx}{x}$$

$$\frac{dx}{x} = \frac{\cos v}{\sin v} dv$$ $\Rightarrow \sin v = t$, $\cos v\, dv = dt$

$$\frac{dx}{x} = \frac{dt}{t}$$

Integrating both sides

$$\int\frac{dx}{x} = \int\frac{dt}{t} + \log c$$

$$\log x = \log t + \log c$$

$$\log x = \log \sin v + \log c$$

$$\log x = \log \sin \frac{y}{x} + \log c$$

$$x = \sin \frac{y}{x} \cdot c$$

$$\frac{x}{c} = \sin \frac{y}{x}$$

$$kx = \sin \frac{y}{x} \quad \left(\because \frac{1}{c} = k\right)$$

Exp. Solve $\frac{dy}{dx} = \frac{x+2y-3}{2x+y-3}$

Soln:- Given the Eqn with compare

$$\frac{dy}{dx} = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$$

$a_1 = 1 \quad b_1 = 2 \quad \& \ c_1 = -3$

$a_2 = 2 \quad b_2 = 1 \quad \& \ c_2 = -3$

Thus $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$ i.e $\frac{1}{2} \neq \frac{2}{1}$

So Case-1 of reducible to homogeneous form then putting

$$x = u+h$$
$$y = v+k$$

h, k are constants

So that

$$\frac{dv}{du} = \frac{(u+h)+2(v+k)-3}{2(u+h)+(v+k)-3}$$

$$\frac{dv}{du} = \frac{u+2v+(h+2k-3)}{2u+v+(2h+k-3)} \quad \text{—— (i)}$$

Choosen $h+2k-3=0$ —— (ii)

$2h+k-3=0$ —— (iii)

from (ii) − 2(iii)

$-3h+3=0 \Rightarrow h=1$ and $k=1$

then Eqn (i)

$$\frac{dv}{du} = \frac{u+2v}{2u+v} \quad \text{—— (a)}$$

Which is homogeneous Eqn.

put $v = tu$ then $\frac{dv}{dt} = t + u\frac{dt}{du}$

$$t + u\frac{dt}{du} = \frac{u+2tu}{2u+tu} = \frac{1+2t}{2+t}$$

$$u\frac{dt}{du} = \frac{1+2t}{2+t} - t = \frac{1+\cancel{2t}-\cancel{2t}-t^2}{2+t}$$

$$u\frac{dt}{du} = \frac{1-t^2}{2+t}$$

$$\frac{du}{u} = \frac{2+t}{1-t^2}dt$$

$$\frac{du}{u} = \frac{2}{1-t^2}dt + \frac{t}{1-t^2}dt$$

$$\frac{du}{u} = \frac{2\,dt}{1-t^2} + \frac{t}{1-t^2}\,dt$$

Integrating both sides.

$$\log u = \log\frac{1+t}{1-t} - \frac{1}{2}\log(1-t^2) + \log c$$

$$\log\frac{u}{c} = \log\frac{1+\frac{v}{u}}{1-\frac{v}{u}} - \frac{1}{2}\log\left(1-\frac{v^2}{u^2}\right)$$

$$\log\frac{u}{c} = \log\left(\frac{u+v}{u-v}\right) - \log\frac{\sqrt{u^2-v^2}}{u}$$

$$= \log\frac{(u+v)}{(u-v)}\cdot\frac{u}{\sqrt{u^2-v^2}} \Rightarrow \log\frac{u\sqrt{u+v}}{(u-v)^{1/2}}$$

$$\frac{u}{c} = \frac{u(\sqrt{u+v})}{\sqrt{u-v}}$$

$$\frac{(x-1)}{c} = \frac{(x-1)\sqrt{x-1+y-1}}{\sqrt{x-1-y+1}}$$

$$\sqrt{x-y} = c\sqrt{x+y-2}$$

$$x-y = c^2(x+y-2)$$