Exp. Examine the convergence of the series

$$\sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5 \cdots\cdots (2n-1)}{2\cdot 4\cdot 6 \cdots\cdots 2n} \cdot \frac{x^{2n+1}}{2n+1}, \ (x>0)$$

Solution:- We have

$$a_n = \frac{1\cdot 3\cdot 5 \cdots\cdots (2n-1)}{2\cdot 4\cdot 6 \cdots\cdots 2n} \cdot \frac{x^{2n+1}}{2n+1}$$

$$a_{n+1} = \frac{1\cdot 3\cdot 5 \cdots\cdots (2n-1)(2n+1)}{2\cdot 4\cdot 6 \cdots\cdots 2n\cdot(2n+2)} \cdot \frac{x^{2n+3}}{2n+3}$$

Then $\because \dfrac{a_n}{a_{n+1}} = \dfrac{(2n+2)(2n+3)}{(2n+1)(2n+1)} \cdot \dfrac{1}{x^2}$

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to\infty} \frac{1}{x^2} \frac{(2n+2)(2n+3)}{(2n+1)(2n+1)}$$

$$= \lim_{n\to\infty} \frac{1}{x^2} \frac{(2+2/n)(2+3/n)}{(2+1/n)(2+1/n)}$$

$$= \frac{1}{x^2} \cdot \frac{(2+0)(2+0)}{(2+0)(2+0)} = \frac{1}{x^2} \cdot \frac{2\times 2}{2\times 2}$$

$$\therefore \lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \frac{1}{x^2}$$

09 Sunday

By Ratio test, $\sum a_n$ converges if $\frac{1}{x^2} > 1$

i.e. $x^2 < 1$ or $x < 1$ (as $x > 0$) and diverges if $x > 1$.

Test fails for $x=1$.

Then $\frac{a_n}{a_{n+1}} = \frac{(2n+2)(2n+3)}{(2n+1)(2n+1)}$

Now

$$n\left(\frac{a_n}{a_{n+1}}-1\right) = n\left[\frac{(2n+2)(2n+3)}{(2n+1)^2}-1\right]$$

$$= n\left[\frac{4n^2+4n+6n+6-4n^2-4n-1}{(2n+1)^2}\right]$$

$$= n\left[\frac{6n+5}{(2n+1)^2}\right]$$

$$= \frac{n^2(6+5/n)}{n^2(2+1/n)^2}$$

$$n\left(\frac{a_n}{a_{n+1}}-1\right) = \frac{6+5/n}{(2+1/n)^2}$$

$$\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}}-1\right) = \lim_{n\to\infty}\frac{(6+5/n)}{(2+1/n)^2}$$

$$= \frac{6}{4} = \frac{3}{2} > 1$$

Hence, $\sum a_n$ converges for $x=1$ by Raabe's Test.