* We have Newton-Raphson formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \text{--- (1)}$$

If we compare Newton-Raphson formula with the relation

$$x_{n+1} = \phi(x_n) \quad \text{--- (2)}$$

of the iterative method (Eqn (2)) with the relation, we obtain

$$\phi(x) = x - \frac{f(x)}{f'(x)}$$

which gives

$$\phi'(x) = \frac{f(x)\, f''(x)}{[f'(x)]^2} \quad \text{--- (3)}$$

To examine the convergence we assume that $f(x)$, $f'(x)$ and $f''(x)$ are continuous and bounded on any interval containing the root $x = \xi$ of the epuotion $f(x) = 0$.

If $\xi$ is a simple root, then $f'(x) \neq 0$. Further since $f'(x)$ is continuous, $|f'(x)| \geq \epsilon$ for some $\epsilon > 0$ in a suitable neighbourhood of $\xi$. Within this neighbourhood we can select an interval such that $|f(x) f''(x)| < \epsilon^2$ and this is possible since $f(\xi) = 0$ and since $f(x)$ is continuously twice differentiable. Hence, in this interval we have

$$|\phi'(x)| < 1 \quad \text{—— } (4)$$

Therefore, Newton-Raphson formula given in epuotion (1) $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ converges, provided that the initial approximation $x_0$ is chosen sufficiently close to $\xi$. When $\xi$ is a multiple root, the Newton-Raphson method still converges but slowly. Convergence can, however, be made faster by modifying Epuotion of Newton-Raphson formula.

To obtain the rate of convergence of the method, we note that $f(\xi) = 0$ so that Taylor's expansion gives

$$f(x_n) + (\xi - x_n) f'(x_n) + \frac{1}{2}(\xi - x_n)^2 f''(x_n) + \cdots + \cdots \frac{1}{\lfloor n}(\xi - x_n)^n f^n(x_n) = 0$$

From which we obtain

$$-\frac{f(x_n)}{f'(x_n)} = (\xi - x_n) + \frac{1}{2}(\xi - x_n)^2 \frac{f''(x_n)}{f'(x_n)} \quad \text{—— } (5)$$

From Eq's (1) & (5)

$$x_{n+1} - \xi = \frac{1}{2}(x_n - \xi)^2 \frac{f''(x_n)}{f'(x_n)} \quad \text{—— (6)}$$

Setting $\quad \epsilon_n = x_n - \xi$

Equation (6) gives

$$\epsilon_{n+1} = \frac{1}{2}\epsilon_n^2 \frac{f''(\xi)}{f'(\xi)} \quad \text{—— (7)}$$

So that the Newton-Raphson process has a second-order or quadratic convergence.