Exp. Examine for convergence or divergence of the series.

$1+2+3+\cdots+n+\cdots\infty$

Solution Give series

$$\sum a_n = 1+2+3+\cdots+n+\cdots\infty$$

Here $S_n = 1+2+3+\cdots+n$

$$S_n = \frac{n}{2}(n+1)$$

$$\lim_{n\to\infty} S_n = \frac{1}{2}\lim_{n\to\infty} n(n+1) \to \infty$$

Hence this series is divergent.

Exp. Show that the geometric series

$1+r+r^2+r^3+\cdots+r^{n-1}+\cdots\infty \quad (r>0)$

(i) converges if $|r|<1$

(ii) diverges if $r\geq 1$

and (iii) oscillates if $r\leq -1$

Solution - We have

$$S_n = 1+r+r^2+\cdots+r^{n-1}$$

$$S_n = \frac{1-r^n}{1-r} = \frac{1}{1-r} - \frac{r^n}{1-r}$$

(I) When $|r|<1$, $\lim_{n\to\infty} r^n = 0$

Also,

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\frac{1}{1-r} - \frac{r^n}{1-r}\right)$$

$$= \lim_{n\to\infty}\left(\frac{1}{1-r}\right) - \lim_{n\to\infty}\frac{r^n}{1-r}$$

$$= \frac{1}{1-r} - \frac{1}{1-r}\lim_{n\to\infty} r^n$$

$$\lim_{n\to\infty} S_n = \frac{1}{1-r} \quad \left(\because \lim_{n\to\infty} r^n = 0\right)$$

Hence $\langle S_n\rangle$ is a convergent sequence geometric series and so the given is convergent.

(II) When $r>1$, $\lim_{n\to\infty} r^n = \infty$

and $S_n = \frac{r^n - 1}{r-1}$ for $r>1$

$$S_n = \frac{r^n}{r-1} - \frac{1}{r-1}$$

So $\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left\{\frac{r^n}{r-1} - \frac{1}{r-1}\right\} \to \infty$

Hence series is divergent because $\langle S_n \rangle$ is a divergent sequence.

(ii) When $r = 1$ then

$$S_n = 1+1+1+1+1 = n$$

$$\lim_{n\to\infty} S_n = \lim_{\to\infty} n = \infty$$

$\therefore$ The series is divergent.

(iii) When (iii) $r = -1$ then series

$$S_n = 1-1+1-1+1-1 \cdots\cdots$$

which is an oscillatory series.

When (iv) $r < -1$, let $r = -p$ such that $p > 1$

Then $r^n = (-1)^n p^n$

$$S_n = \frac{1-r^n}{1-r} \quad \text{for } r < 1$$

putting r^n value

$$S_n = \frac{1-(-1)^n p^n}{1+p} \quad \text{as} \lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(\frac{1-(-1)^n p^n}{1+p}\right) \to \infty$$

$\because \lim_{n\to\infty} p^n = \infty$

Hence the series is oscillates.