Exact systems:

A system of differential equations

$$\sum_{j=1}^{n+1} m_{ij}(y_1, y_2, \cdots, y_{n+1})\,dy_j = 0 \quad i = 1,2,3,\cdots n$$

is said to be exact system if

$$\frac{\partial m_{ij}}{\partial y_{j+k}} = \frac{\partial m_{ij+k}}{\partial y_j} \qquad i,j=1,\cdots n; \quad k_j = 1 \cdots (n+1-j).$$

The solution of this system is

$$\sum_{j=1}^{n+1} \int_{y_{j0}}^{y_j} m_{ij}(y_1, \cdots y_{j-1}, s, y_{j+1\,0}, \cdots, y_{n+1\,0})\,ds = C_i$$

$$i = 1,2,\cdots,n.$$

where y_{j0} are any convenient constants.

Exp. Solve

$$(1+y_2+y_3)dy_1 + (1+y_1+y_3)dy_2 + (1+y_1+y_2)dy_3 = 0,$$

$$y_2y_3\,dy_1 + y_1y_3\,dy_2 + y_1y_2\,dy_3 = 0$$

Sol. We have $n=2$

$m_{11} = 1+y_2+y_3$, $m_{12} = 1+y_1+y_3$, $m_{13} = 1+y_1+y_2$

$m_{21} = y_2y_3$, $m_{22} = y_1y_3$, $m_{23} = y_1y_2$

Since $\dfrac{\partial m_{11}}{\partial y_2} = \dfrac{\partial m_{12}}{\partial y_1} = 1$

$$\frac{\partial m_{12}}{\partial y_3} = \frac{\partial m_{13}}{\partial y_2} = 1$$

$$\frac{\partial m_{21}}{\partial y_2} = \frac{\partial m_{22}}{\partial y_1} = y_3$$

$$\frac{\partial m_{22}}{\partial y_3} = \frac{\partial m_{23}}{\partial y_2} = y_1$$

Hence; the given system is found to be exact, thus putting $y_{10} = y_{20} = y_{30} = 0$ we get

$$\int_0^{y_1} ds + \int_0^{y_2} (1+y_1)\,ds + \int_0^{y_3} (1+y_1+y_2)\,ds$$

$$= y_1 + (1+y_1)y_2 + (1+y_1+y_2)y_3 = c_1$$

$$= y_1 + y_2 + y_1y_2 + y_3 + y_1y_3 + y_2y_3 = c_1$$

$$= y_1 + y_2 + y_3 + y_1y_2 + y_1y_3 + y_2y_3 = c_1$$

and $$\int_0^{y_3} y_1y_2\,ds = y_1y_2y_3 = c_2$$

Then the resulting solution is given by the following system

$$y_1 + y_2 + y_2 + y_1y_2 + y_1y_3 + y_2y_3 = c_1$$

$$y_1y_2y_3 = c_2$$