* Euler-Maclaurin formula:-

Let the expansion of $1/(e^x-1)$ in ascending powers of x, obtained by writing the Maclaurin expansion of e^x and simplifying

$$\frac{1}{e^x-1} = \frac{1}{x} - \frac{1}{2}x^0 + B_1 x + B_3 x^3 + B_5 x^5 + \cdots \quad (1)$$

Where $B_{2r} = 0$, $B_1 = \frac{1}{12}$, $B_3 = \frac{1}{720}$, $B_5 = \frac{1}{30240}$ .. etc

If we set $x = hD$ in Eqn (1) and use the relation $E = e^{hD}$, then we obtain the idenity

$$\frac{1}{E-1} = \frac{1}{hD} - \frac{1}{2} + B_1 hD + B_3 h^3 D^3 + B_5 h^5 D^5 + \cdots \quad (2)$$

Or equivalently

$$\frac{E^n-1}{E-1} = \frac{1}{hD}(E^n-1) - \frac{1}{2}(E^n-1) + B_1 hD(E^n-1)$$
$$+ B_3 h^3 D^3 (E^n-1) + \cdots \quad (3)$$

Operating this identity on y_0, we obtain

$$\left(\frac{E^n-1}{E-1}\right) y_0 = \frac{1}{hD}(E^n-1)y_0 - \frac{1}{2}(E^n-1)y_0 + B_1 hD(E^n-1)y_0$$
$$+ B_3 h^3 D^3 (E^n-1) y_0 + \cdots$$

$$\frac{(E^n-1)}{E-1} y_0 = \frac{1}{hD}(y_n - y_0) - \frac{1}{2}(y_n - y_0) + B_1 h(y_n' - y_0')$$
$$+ B_3 h^3 (y_n''' - y_0''') + B_5 h^5 (y_n^v - y_0^v) \quad (4)$$

The left hand side denotes the sum $y_0 + y_1 + y_2 + \cdots + y_{n-1}$, whereas the term

$$\frac{1}{hD}(y_n - y_0)$$

$$\frac{1}{h}\int_{x_0}^{x_n} y\,dx \text{ , since } \frac{1}{D} \text{ as integration}$$

Hence, Eqn (4) becomes

$$\int_{x_0}^{x_n} y\,dx = \frac{h}{2}(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n) + \left(-\frac{h^2}{12}\right)(y'_n - y'_0) + \frac{h^4}{720}(y'''_n - y'''_0) + \left(-\frac{h^6}{30,240}\right)(y^v_n - y^v_0) + \cdots \quad (5)$$

Which is called the Euler-Maclaurin's formula for integration.

Exp. Evaluate $I = \int_0^{\pi/2} \sin x\,dx$ using Euler-Maclaurin's formula.

<u>Solution</u> - Using Euler-Maclaurin's formula

$$\int_{x_0}^{x_n} y\,dx = \frac{h}{2}[y_0 + 2y_1 + 2y_2 + \cdots 2y_{n-1} + y_n) - \frac{h^2}{12}(y'_n - y'_0) + \frac{h^4}{720}(y'''_n - y'''_0) - \frac{h^6}{30240}(y^v_n - y^v_0) + \cdots \quad (1)$$

In this case formula simplifies to

$$\int_0^{\pi/2} \sin x\,dx = \frac{h}{2}(y_0 + 2y_1 + 2y_2 + \cdots 2y_{n-1} + y_n) + \frac{h^2}{12} + \frac{h^4}{720} + \frac{h^6}{30,240} + \cdots \quad (2)$$

To evaluate the integral, we take $h = \pi/4$

then we obtain from equation (2).

The value of x and y from $h = \pi/4$

	x	$y = \sin x$	
x_0	0	$\sin 0 = 0$	y_0
x_1	$\pi/4$	$\sin \pi/4 = \frac{1}{\sqrt{2}}$	y_1
x_2	$2\pi/4 = \pi/2$	$\sin \pi/2 = 1$	y_2

$$\int_0^{\pi/2} \sin x = \frac{\pi}{4\times 2}\left[0 + 2\times\frac{1}{\sqrt{2}} + 1\right] + \frac{\pi^2}{192} + \frac{\pi^4}{184320} +$$

$$= \frac{\pi}{8}\left[1 + \sqrt{2}\right] + \frac{\pi^2}{192} + \frac{\pi^4}{184320}, \text{ approximately}$$

$$= \frac{3.142 \times 2.414}{8} + 0.051446 + 0.000529$$

$$= 0.948441 + 0.051446 + 0.000529$$

$$\int_0^{\pi/2} \sin x = 1.000416$$

By Direct definite integral

$$\int_0^{\pi/2} \sin x\,dx = \left(-\cos x\right)_0^{\pi/2}$$

$$= \left[-\cos \pi/2 + \cos 0\right]$$

$$\left[0 + 1\right] = 1$$

$$\int_0^{\pi/2} \sin x\,dx = 1$$

On other hand $h = \pi/8$ then the the value of x and y.

x	$y = \sin x$	
0	$\sin 0 = 0.0$	y_0
$\pi/8$	$\sin \pi/8 = 0.382499$	y_1
$2\pi/8$	$\sin 2\pi/8 = 0.706826$	y_2
$3\pi/8$	$\sin 3\pi/8 = 0.923651$	y_{n-1}
$4\pi/8$	$\sin 4\pi/8 = 1.000$	y_n

$$\int_0^{\pi/2} \sin x \, dx = \frac{\pi}{8} \cdot \frac{1}{2} \left[0 + 2\times(0.382499) + 2\times 0.706826 + 2\times(0.923651) + 1.0 \right]$$

$$+ \frac{\pi^2}{64 \times 12} + \frac{\pi^4}{(8)^4 \times 720}$$

$$= 0.986393 + 0.12851 + 0.000033$$

$$\int_0^{\pi/2} \sin x \, dx = 1.000003$$

$$\simeq 1$$