Euler's Method:-

Consider the equation

$$\frac{dy}{dx} = f(x, y) \text{ given that } y(x_0) = y_0 \quad \text{——— (1)}$$

Suppose that we wish to solve the equation (1) for values of y at $x = x_i = x_0 + ih$ $(i = 1, 2, \cdots n)$
Integrating Eqn (1), we obtain

$$y_1 = y_0 + \int_{x_0}^{x_1} f(x, y)\,dx \quad \text{——— (2)}$$

Assuming that $f(x, y) = f(x_0, y_0)$ in $x_0 \le x \le x_1$, this gives Euler's formula

$$y_1 = y_0 + hf(x_0, y_0) \text{ — (3)} \quad (\because h = x_1 - x_0)$$

Similarly for the range $x_1 \le x \le x_2$, we have

$$y_2 = y_1 + \int_{x_1}^{x_2} f(x, y)\,dx$$

substituting $f(x_1, y_1)$ for $f(x, y)$ in $x_1 \le x \le x_2$ we obtain

$$y_2 = y_1 + hf(x_1, y_1) \quad \text{——— (4)}$$

Proceeding in this way, we obtain general formula

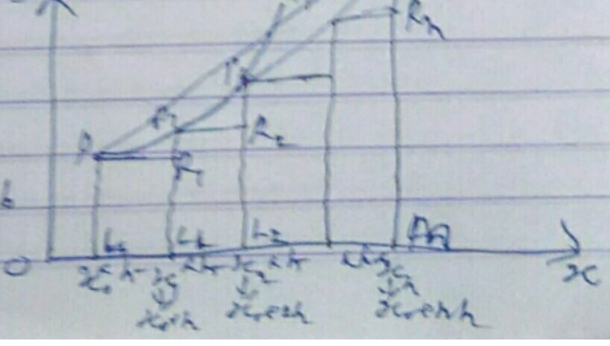
$$y_{n+1} = y_n + hf(x_n, y_n) \quad n = 0, 1, 2, \cdots \quad \text{——— (5)}$$

The process is very slow and to obtain reasonable accuracy with Euler's method, we need to take a smaller value for h.

Another Method for this

From Eqn (1). Its curve of solution through $P(x_0, y_0)$ is shown denoted in the figure

Now, we have find the ordinate of any point Q on this curve.
Let us divide LM into n-sub intervals each of width h at L_1



In the interval LL_1, we approximate the curve by the tangent at P. If the ordinate through L_1 meets this tangent in $P_1(x_0+h, y_1)$ or $P_1(x_1, y_1)$ then

$$y_1 = L_1P_1 = LP + R_1P_1 = y_0 + PR_1 \tan\theta$$

$$= y_0 + h\left(\frac{dy}{dx}\right)_P = y_0 + hf(x_0, y_0) \quad \because PR_1 = h$$

Let P_1Q_1 be the curve of solution of Eqn (1) through P_1 and let its tangent at P_1 meet the ordinate through L_2 in $P_2(x_0+2h, y_2)$ or $P_2(x_2, y_2)$ then

$$y_2 = y_1 + hf(x_1, y_1)$$

Repeating this process n times, then

$$y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$$

where $x_{n-1} = x_0 + (n-1)h$

and $y_{n+1} = y_n + hf(x_n, y_n)$

This is Euler's method of finding an approximation solution of (1).

Exp. Using Euler's method, find a approximate value of y corresponding to $x=1$, given that $\frac{dy}{dx}=x+y$ and $y=1$ at $x=0$

Solution We take $n=10$ and $h=0.1$ which is small. Then.

x	y	$f(x,y)=x+y=dy/dx$	old $y + hf(x,y)$ = new y
0.0	1.0	0.0+1.0 = 1.00	1.00 + 0.1 (1.00) = 1.10
0.1	1.1 new y	0.1+1.1 = 1.20	1.10 + 0.1 (1.20) = 1.22
0.2	1.22	0.2+1.22 = 1.42	1.22+0.1 (1.42) = 1.36
0.3	1.36	0.3+1.36 = 1.66	1.36+0.1 (1.66) = 1.53
0.4	1.53	0.4+1.53 = 1.93	1.53 +0.1 (1.93) = 1.72
0.5	1.72	0.5+1.72 = 2.22	1.72+0.1 (2.22) = 1.94
0.6	1.94	0.6 + 1.94 = 2.54	1.94+0.1 (2.54) = 2.19
0.7	2.19	0.7 + 2.19 = 2.89	2.19+0.1 (2.89) = 2.48
0.8	2.48	0.8 + 2.48 = 3.28	2.48 +0.1 (3.28) = 2.81
0.9	2.81	0.9 + 2.81 = 3.71	2.81 +0.1 (3.71) = 3.18
1.0	3.18	1.0 + 3.18 = 4.18	3.18+0.1 (4.18) = 3.60

Thus the required approximate value of

$y = 3.18$