Embedding of ring without unity

Q. Prove that any ring without unity can be embedded in a ring with unity

Let R be a ring without unity. Let Z be the ring of integers. We form the cartesian product $R \times Z$ as follows:

$$R \times Z = \{(a, m) : a \in R,\ m \in Z\}$$

Now by suitable definitions of addition & multiplication we shall make $R \times Z$ a ring which has unity & which contains a subring isomorphic to R.

Let $(a, m), (b, n) \in R \times Z$

We define $(a, m) + (b, n) = (a+b,\ m+n)$

& $(a, m)(b, n) = (ab + na + mb,\ mn)$

Clearly the sum & product of two elements of $R \times Z$ are the elements of $R \times Z$

Hence $R \times Z$ is closed under addition & multiplication.

Associative law for addition

Let $(a, m), (b, n), (c, p) \in R \times Z$

Then $[(a, m) + (b, n)] + (c, p)$

$= (a+b,\ m+n) + (c, p)$

$= [(a+b) + c,\ (m+n) + p]$

$= [a + (b+c),\ m + (n+p)]$

$= (a, m) + (b+c,\ n+p)$

$= (a, m) + [(b, n) + (c, p)]$

Hence associative law for addition holds.

Existence of additive identity

There exists an additive identity $(0, 0) \in R \times Z$

Such that for every $(a, m) \in R \times Z$

$(a, m) + (0, 0) = (a+0,\ m+0) = (a, m) = (0, 0) + (a, m)$

Existence of additive inverse

For every $(a, m) \in R \times Z$ there exists additive inverse $(-a, -m)$ such that

$$(a, m) + (-a, -m) = [a + (-a),\ m + (-m)]$$
$$= (0, 0)$$
$$= (-a, -m) + (a, m)$$

Commutative law for addition

$$(a, m) + (b, n) = (a+b,\ m+n)$$
$$= (b+a,\ n+m)$$
$$= (b, n) + (a, m)$$

i.e. commutative law for addition holds.

Associative law for multiplication

$$[(a,m)(b,n)](c,p) = (ab + na + mb,\ mn)(c, p)$$
$$= [(ab + na + mb)c + p(ab + na + mb) + (mn)c,\ (mn)p]$$
$$= (abc + nac + mbc + pab + pna + pmb + mnc,\ mnp)$$

$$\& \ (a,m)[(b,n)(c,p)] = (a,m)(bc + pb + nc,\ np)$$
$$= [a(bc + pb + nc) + (np)a + m(bc + pb + nc),\ m(np)]$$
$$= (abc + apb + anc + npa + mbc + mpb + mnc,\ mnp)$$
$$= (abc + nac + mbc + pab + pna + pmb + mnc,\ mnp)$$

$$\therefore \ [(a,m)(b,n)](c,p) = (a,m)[(b,n)(c,p)]$$

i.e. associative law for multiplication holds.

Distributive laws:

$$(a,m)[(b,n) + (c,p)] = (a,m)(b+c,\ n+p)$$
$$= [a(b+c) + (n+p)a + m(b+c),\ m(n+p)]$$
$$= (ab + ac + na + pa + mb + mc,\ mn + mp)$$

$$\& \ (a,m)(b,n) + (a,m)(c,p)$$
$$= (ab + na + mb,\ mn) + (ac + pa + mc,\ mp)$$
$$= (ab + na + mb + ac + pa + mc,\ mn + mp)$$
$$= (ab + ac + na + pa + mb + mc,\ mn + mp)$$

$$\therefore \ (a,m)[(b,n) + (c,p)] = (a,m)(b,n) + (a,m)(c,p)$$

i.e. distributive laws hold.

Hence $R \times Z$ is a ring.

Existence of unity element

There exists an unity element $(0,1) \in R \times Z$ such that

$$(a, m)(0,1) = (a0 + 1a + m0, m1)$$
$$= (a, m)$$
$$= (0,1)(a, m)$$

Let us now consider the subset $R \times \{0\}$ of $R \times Z$

This is a subring of $R \times Z$ because if $(a,0), (b,0) \in R \times \{0\}$,

then $(a,0) + \{-(b,0)\} = (a,0) + (-b,0)$
$$= [a+(-b), 0+0]$$
$$= (a-b, 0) \in R \times \{0\}$$

& $(a,0)(b,0) = (ab + 0a + 0b, 0)$
$$= (ab, 0) \in R \times \{0\}$$

Let us consider a mapping $\phi : R \to R \times \{0\}$ such that

$\phi(a) = (a, 0) \; \forall \; a \in R$

Clearly ϕ is one-one onto mapping

Also ϕ is a homomorphism because

$$\phi(a+b) = (a+b, 0) = (a,0) + (b,0) = \phi(a) + \phi(b)$$
$$\& \; \phi(ab) = (ab, 0) = (a,0)(b,0) = \phi(a)\phi(b)$$

Thus ϕ is an isomorphism i.e. R is isomorphic to $R \times \{0\}$

Hence R is embedded in $R \times Z$ which is a ring with unity.

i.e. a ring without unity can be embedded in a ring with unity.