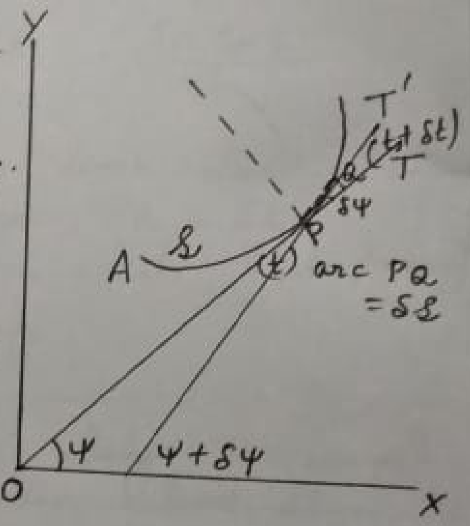
DYNAMICS

1 (a) Find expression for the tangential and normal accelerations of a particle moving in a plane curve.

Let P, Q be the position of the particle at time t and $t+\delta t$ respectively.



Let A be a fixed point on the path of the particle.
Let arc $AP = s$,

arc $AQ = s + \delta s$

so that arc $PQ = \delta s$

Let PT be the tangent to the path at P and make angle ψ with X-axis. Let QT' be the tangent at Q and make angle between these tangents is $\delta\psi$.

join the chord PQ and let QM be perpendicular to PT.

Let $\angle MPQ = \epsilon$

we first find expression for the tangential and normal velocities at P.

The tangential velocity at P is

$$v = \lim_{\delta t \to 0} \frac{\text{displacement along the tangent at P in time } \delta t}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{PM}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{\text{chord } PQ \cos \epsilon}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{\text{chord } PQ}{\delta s} \cdot \frac{\delta s \cos \epsilon}{\delta t}$$

$= \frac{ds}{dt}$ [$\because$ in the limit chord PQ $=$ arc $PQ = \delta s$ and $\epsilon < \delta\psi$, which tends to zero]

Normal velocity at P

$$= \lim_{\delta t \to 0} \frac{\text{displacement along the normal at } P \text{ in limit } \delta t}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{\text{chord } PQ \sin \epsilon}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{\text{chord } PQ}{\delta t} \cdot \frac{\delta t}{\delta t} \cdot \sin \epsilon$$

$= 0$ [$\because \epsilon = 0$ in the limit]

We now obtain expression for acceleration

tangential acceleration at P

$$= \lim_{\delta t \to 0} \frac{\text{increase in velocity along the tangent at } P \text{ in time } \delta t}{}$$

$= \lim_{\delta t \to 0} \frac{(v + \delta v) \cos \delta\psi - v}{\delta t}$ (where $v + \delta v$ is the tangential velocity at Q)

$$= \lim_{\delta t \to 0} \frac{(v+\delta v)\left[1 - \frac{(\delta\psi)^2}{\lfloor 2} + \cdots\right] - v}{\delta t}$$

$$= \lim_{\delta t \to 0} \left[\frac{\delta v}{\delta t} - \frac{(v+\delta v)\frac{(\delta\psi)^2}{\lfloor 2} + \cdots}{\delta t}\right]$$

$$= \frac{dv}{dt}$$

$$= \frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{d^2s}{dt^2}$$

Since $\frac{dv}{dt} = \frac{dv}{ds}\frac{ds}{dt} = \frac{dv}{ds}v$,

we find that the tangential acceleration at P may be expressed as $\frac{dv}{dt}$ or $\frac{d^2s}{dt^2}$.

<u>Normal acceleration at P</u>

$$= \lim_{\delta t \to 0} \frac{\text{increase in velocity along the normal at P in lim } \delta t}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{(v+\delta v)\sin\delta\psi - 0}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{(v+\delta v)\left(\delta\psi - \frac{(\delta\psi)^3}{\lfloor 3} + \cdots\right)}{\delta t}$$

$$= \lim_{\delta t \to 0} \left[\frac{v\,\delta\psi}{\delta t} + \frac{\delta v\,\delta\psi}{\delta t} - \cdots\right]$$

$$= v\frac{d\psi}{dt}$$

$$= v\frac{d\psi}{ds}\frac{ds}{dt}$$

$$= v \cdot \frac{1}{\rho} \cdot v$$

$= \frac{v^2}{\rho}$, where $\rho = \frac{ds}{d\psi}$ is the radius of curvature of the path at P.