(4) A particle moves in a curve $y = a\log\sec\left(\frac{x}{a}\right)$ in such a way that the tangent to the curve rotates uniformly. Prove that the resultant acceleration of the particle varies as the square of the radius of curvature.

The given curve is $y = a\log\sec\left(\frac{x}{a}\right)$ ——— (1)

Differentiating w.r.t. x, we get $\tan\psi = \frac{dy}{dx}$

$$= \frac{a\sec\left(\frac{x}{a}\right)}{\sec\frac{x}{a}}\tan\left(\frac{x}{a}\right)\frac{1}{a} = \tan\frac{x}{a}$$

$$\therefore \psi = \frac{x}{a} \quad \text{——— (2)}$$

By question $\frac{d\psi}{dt} = \text{Constant} = k$ (say)

$$\therefore \frac{1}{a}\frac{dx}{dt} = k$$

$$\text{or, } \frac{d^2x}{dt^2} = 0 \quad \text{——— (3)}$$

From (1) and (2) we have $y = a\log\sec\psi$

$$\therefore \frac{dy}{dt} = \frac{a}{\sec\psi}\sec\psi\tan\psi\frac{d\psi}{dt} = ak\tan\psi$$

$$\therefore \frac{d^2y}{dt^2} = ak\sec^2\psi\frac{d\psi}{dt}$$

$$= ak^2\sec^2\psi \quad \text{——— (4)}$$

$\therefore$ (Resultant acceleration$)^2 = \left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2$

$$= (ak^2 \sec^2 \psi)^2 \quad \text{—— (5)}$$

But the radius of curvature is given by $\rho = \frac{(1+\tan^2\psi)^{3/2}}{\frac{d^2y}{dx^2}} = \frac{\sec^3\psi}{\sec^2\psi} = \sec\psi$

Hence (5) gives

resultant acceleration $= ak^2 \sec^2\psi$

$$= ak^2\rho^2 \propto \rho^2$$