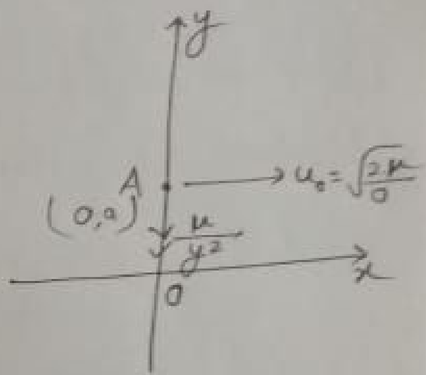
DYNAMICS

Motion in a plane - problems

1. A particle moves in a plane with an acceleration μy^{-2} parallel to y-axis and directed towards x-axis. If it is projected from a point distance a from the x-axis with a velocity $\sqrt{\frac{2\mu}{-2a}}$ parallel to x-axis show that its path is a cycloide.



Let the point of projection be taken on the y-axis at a distance a from the x-axis.

Then the equation of motion is

$$\frac{d^2x}{dt^2} = 0 \quad \text{--- (1)}$$

$$\frac{d^2y}{dt^2} = \frac{-\mu}{y^2} \quad \text{--- (2)}$$

The initial conditions are $t=0$: $x=a$,

$$y = a, \quad \frac{dx}{dt} = \sqrt{\frac{2\mu}{a}}$$

and $\frac{dy}{dt} = 0$

Integrating (1) we get $\frac{dx}{dt} = c_1$ (= constant)

Applying the initial condition, we get

$$c_1 = \sqrt{\frac{2\mu}{a}} \quad \text{--- (3)}$$

Equation (2) can be written as

$$v\frac{dv}{dy} = -\frac{\mu}{y^2} \quad \text{where } v = \frac{dy}{dt}$$

Integrating, we get $\frac{v^2}{2} = \frac{\mu}{y} + C_2$

Applying the initial condition, we get

$$0 = \frac{\mu}{a} + C_2$$

Hence $v^2 = 2\mu\left(\frac{1}{y} - \frac{1}{a}\right)$

$$\therefore \frac{dy}{dt} = -\sqrt{\frac{2\mu}{a}}\sqrt{\frac{a-y}{y}} \quad \text{—— (4)}$$

Eliminating t between (3) and (4) we get

$$\frac{dy}{dx} = -\sqrt{\frac{a-y}{y}}$$

$$\text{or, } dx = -\sqrt{\frac{y}{a-y}}\,dy$$

Integrating $x + C_3 = -\int \sqrt{\frac{y}{a-y}}\,dy$

putting $y = a\cos^2\theta$, this gives

$$x + C_3 = \int \frac{\cos\theta}{\sin\theta} \cdot 2a\cos\theta\sin\theta\, d\theta$$

$$\text{or, } x + C_3 = a\left\{\theta + \frac{\sin 2\theta}{2}\right\}$$

Applying the initial condition that $x = 0$ when $y = a$, i.e. $\theta = 0$.

we get $C_3 = 0$.

Hence the path is given by

$$x = \frac{a}{2}(2\theta + \sin 2\theta)$$

$$y = \frac{a}{2}(2 + \cos 2\theta)$$

This is cycloid.