DYNAMICS

Remaining part of Q. 9

DEDUCTION OF KEPLER'S LAWS OF PLANETORY MOTION FROM NEWTON'S LAW OF GRAVITATION :-

Newtons law of gravitation states that every particle of matter attracts every other particle of matter with a force $F = G\frac{Mm}{r^2}$,

where M, m are the masses of the two particles r is the distance between them and G is the constant of gravitation.

Hence the sun of mass M attracts a planet of mass m with a force

$$F = \frac{m\mu}{r^2} \quad \text{where } \mu = GM = \text{constant}$$

Thus the law of gravitation is the inverse square law.

Now it is Known that in a central field of force varying inversely as the square of distance the path of a particle is closed curve.

If (r, θ) be the polar coordinates of a planet of mass m at time t, with the sun S as the pole, the differential equation of the path, compatible with the law of gravitation, is

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = \frac{-\mu}{r^2} \quad \text{———} (1)$$

and $\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = 0$ —— (2)

because the radial acceleration of the planet is $\frac{\mu}{r^2}$ and is directed towards to pole S (sun) and the transverse acceleration is zero.

equation (2). gives

$$r^2\frac{d\theta}{dt} = \text{constant} = h \text{ (say)} \text{ —— (3)}$$

or, $\frac{d\theta}{dt} = hu^2$ where $r = \frac{1}{u}$

$$\therefore \frac{dr}{dt} = \frac{dr}{du}\frac{du}{d\theta}\frac{d\theta}{dt}$$

$$= -\frac{1}{u^2}\frac{du}{d\theta}\cdot hu^2$$

$$= -h\frac{du}{d\theta}$$

$$\therefore \frac{d^2r}{dt^2} = -h\frac{d^2u}{d\theta^2}\frac{d\theta}{dt}$$

$$= -h\frac{d^2u}{d\theta^2}\cdot hu^2$$

$$= -h^2u^2\frac{d^2u}{d\theta^2}$$

Hence (1) becomes

$$-h^2u^2\frac{d^2u}{d\theta^2} - \frac{1}{u} - h^2u^4 = -\mu u^2$$

$$h^2u^2\left[\frac{d^2u}{d\theta^2} + u\right] = -\mu u^2$$

or, $\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2}$ —— (4)

This is a second order linear differential equation with constant coefficients.

The A.E. is $(D^2+1) = 0$

or, $D = \pm i$

Hence $C.F. = A\cos(\theta-\omega)$

$(A, \omega = \text{constant})$

Also $P.I. = \frac{1}{1+D^2}\left(\frac{\mu}{h^2}\right)$

$= (1+D^2)^{-1}\frac{\mu}{h^2}$

$= (1-D^2+ \cdots)\left(\frac{\mu}{h^2}\right)$

$= \frac{\mu}{h^2}$

Hence the complete integral of (4) is

$u = A\cos(\theta-\omega) + \frac{\mu}{h^2}$

This can be written as

$u = \frac{\mu}{h^2}[1+e\cos(\theta-\omega)]$ $\left(e = \frac{Ah^2}{\mu} = \text{cons}\right.$

Putting $h^2 = \mu l$ ——— (5)

This becomes $\frac{l}{r} = 1+e\cos(\theta-\omega)$

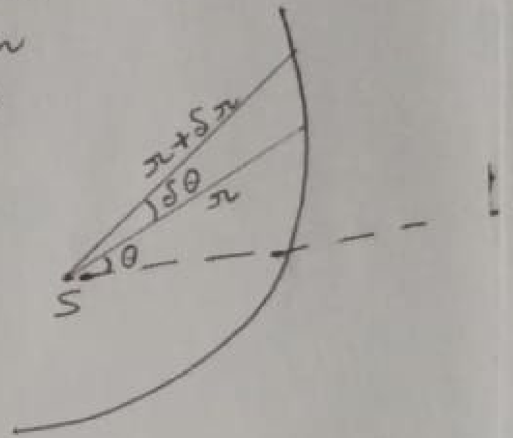
——— (6)

Thus (6) is the polar equation of the path of the planet. Now (6) is a conic with the pole, S as a focus. Also as maintioned above, the path is a closed curve.

Hence (6) i.e. the path of the planet, is an ellipse with S as a focus.

This proves Keplers law I.

The rate of description of area by the radius vector to the planet at (r, θ)



$$= \lim_{\delta t \to 0} \frac{\frac{1}{2} r (r+\delta r) \sin \delta\theta}{\delta t}$$

$$= \frac{1}{2} r^2 \frac{d\theta}{dt}$$

$$= \frac{1}{2} h$$

$= \text{Constant}$. [by (3)] —— (7)

This means that the radius vector from the sun to the planet sweeps out equal area in equal intervals of time.

This proves the second Keplers law (i.e. II).

Lastly if a and b be the semi-major and semi-minor axis of the ellipse (6), its area πab. Hence if T be the periodic time of the planet i.e. if the planet completes one revolution round the sun in time T. then the rate of description of area by a radius vector is $\frac{\pi ab}{T}$.

Hence from 7,

$$\frac{\pi ab}{T} = \frac{1}{2}h = \frac{1}{2}\sqrt{\mu l} \text{, by } 5$$

or, $$T^2 = \frac{4\pi^2 a^2 b^2}{\mu l}$$

$$= \frac{4\pi^2 a^2 b^2}{\mu \cdot \frac{b^2}{a}} \quad \left[\because l = \text{semi lectus rectum} = \frac{b^2}{a}\right]$$

or, $$T^2 = \frac{4\pi^2}{\mu} \cdot a^3$$

This proves Kepler's 3rd law (III)