Q.6 A particle moves in an ellipse under a force which is always directed towards its focus. Find the law of force and velocity at any point.

We know that for a central orbit, the law of force is given by

$$P = h^2 u^2 \left[u + \frac{d^2u}{d\theta^2} \right] \quad \text{——— (1)}$$

is usual notation.

Now the equation to an ellipse referred to a focus as the pole is

$$r = \frac{l}{1 + e\cos\theta}$$

This gives $u = \frac{1}{r} = \frac{1}{l} + \frac{e\cos\theta}{l}$

$$\therefore \frac{du}{d\theta} = \frac{-e\sin\theta}{l}$$

$$\frac{d^2u}{d\theta^2} = \frac{-e\cos\theta}{l}$$

Hence (1) gives

$$P = h^2u^2\left(\frac{1}{l}\right) = \frac{h^2u^2}{l}$$

$$= \frac{h^2}{l} \cdot \frac{1}{r^2}$$

$$= \frac{\mu}{r^2} \left(\text{where } \mu = \frac{h^2}{l}\right)$$

This shows that the acceleration at any point in the elliptical, orbit varies inversely as the square of the distance of the point from the focus of the ellipse.

This is the law of force.

<u>Velocity</u>

we have $v^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right]$

$$= h^2\left[\left(\frac{1}{l} + \frac{e\cos\theta}{l}\right)^2 + \frac{e^2\sin^2\theta}{l^2}\right]$$

$$= \frac{h^2}{l}\left[1 + 2e\cos\theta + e^2\right]$$

$$= \frac{h^2}{l}\left[\frac{2(1+e\cos\theta)}{l} - \frac{1-e^2}{l}\right]$$

$$= \frac{h^2}{l}\left[\frac{2}{r} - \frac{1}{a}\right]$$

[$\because$ If a, b be the semi-axes of the ellipse, then $l = \frac{b^2}{a}$ and $a^2e^2 = a^2 - b^2$ so that $a^2(1-e^2) = b^2 = al$]

This gives velocity at any point.

<u>Remark</u>

If T be the time taken by the particle to describe the whole ellipse once, i.e. if T be the time period of a complete revolution, then

$$\frac{1}{2}hT = \text{area of the ellipse} = \pi ab$$

$$T = \frac{2\pi ab}{h}$$

$$= \frac{2\pi a \cdot \sqrt{al}}{h}$$

$$= \frac{2\pi\sqrt{l}}{h} \cdot a^{3/2}$$

$$= \frac{2\pi a^{3/2}}{\sqrt{\mu}}$$

where, $\mu = \frac{h^2}{l}$

Thus $T^2 \propto a^3$

This shows that the square of time period of one complete revolution varies as the cube of the semi-major axis of the ellipse.