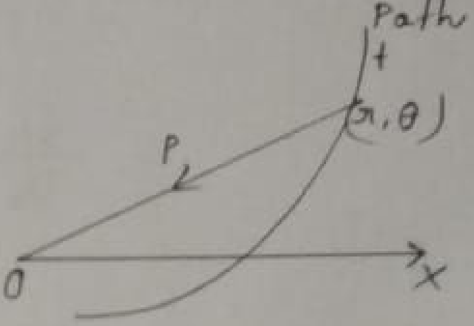
DYNAMICS

Q (2) Find the differential equation of the path of a particle moving in a central field of reciprocal force in polar coordinates.

Let O be the centre of acceleration and with O as the pole let (r, θ) be the polar coordinates of any point on the path of the particle at time t.



Let P be the acceleration at (r, θ) since the acceleration P is directed towards O, and there is no transverse acceleration we have

$$\frac{d^2 r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -P \quad \text{———} (1)$$

and

$$\frac{1}{r}\frac{d}{dt}\left(r^2 \frac{d\theta}{dt}\right) = 0 \quad \text{———} (2)$$

From (2) we get

$$\frac{r^2 \, d\theta}{dt} = \text{Constant} = h \text{ (say)}$$

or $\dfrac{d\theta}{dt} = hu^2$ [where $u = \dfrac{1}{r}$]

Also

$$\frac{dr}{dt} = \frac{d}{dt}\left(\frac{1}{u}\right)$$

$$= -\frac{1}{u^2}\frac{du}{dt}$$

$$= -\frac{1}{u^2}\frac{du}{d\theta}\frac{d\theta}{dt}$$

$$= -\frac{1}{u^2}\frac{du}{d\theta} \cdot hu^2$$

$$= -h\frac{du}{d\theta}$$

$$\frac{d^2r}{dt^2} = -h\frac{d}{d\theta}\left(\frac{du}{d\theta}\right)\frac{d\theta}{dt}$$

$$= -h\frac{du^2}{d\theta^2}\frac{d\theta}{dt}$$

$$= -h\frac{d^2u}{d\theta^2} \cdot hu^2$$

$$= -h^2u^2\frac{d^2u}{d\theta^2}$$

Hence (1) becomes after putting the value of $\frac{d\theta}{dt}$ and $\frac{d^2r}{dt^2}$

$$-h^2u^2\frac{d^2u}{d\theta^2} - \frac{1}{u} \cdot h^2u^4 = -P$$

or, $-h^2u^2\left[u + \frac{d^2u}{d\theta^2}\right] = -P$

or, $\left[\frac{d^2u}{d\theta^2} + u\right] = \frac{P}{h^2u^2}$ ——— (3)

This is required differential equation of the path.