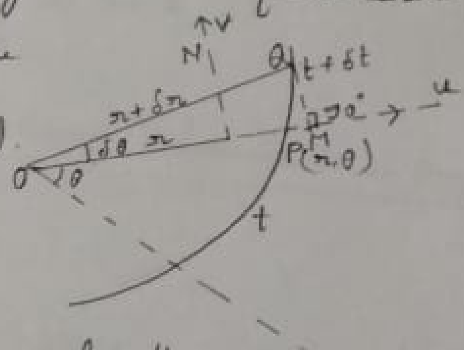
DYNAMICS

Q) Find expression for the radial and transverse acceleration of a particle moving in a plane curve.

We shall first find expressions for the radial and transverse velocities of a particle moving in a plane curve.

Let at any time t, the position of the particle be $P(r, \theta)$. Let at time $t+\delta t$ the particle be at $Q(r+\delta r, \theta+\delta\theta)$



At time t the radial velocity of the particle is along the radius vector OP. Let it be denoted by u.

At the same time t, the transverse velocity of the particle, i.e. velocity normal to OP in θ-increasing direction, is along PM. Let it be denoted by v.

Since the velocity is the time rate of displacement, we have

$$u = \lim_{\delta t \to 0} \frac{\text{displacement in direction OP in time } \delta t}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{PM}{\delta t}, \text{ where M is the foot of perp. from Q on OP.}$$

$$= \lim_{\delta t \to 0} \frac{OM - OP}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{OQ \cos \delta\theta - r}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{(r+\delta r)\left[1 - \frac{(\delta\theta)^2}{\lfloor 2} + \cdots\right] - r}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{\delta r\left[1 - \frac{(\delta\theta)^2}{2} + \cdots\right] - r\left[\frac{(\delta\theta)^2}{2} + \cdots\right]}{\delta t}$$

since $\delta\theta \to 0$ as $\delta t \to 0$.

$$\therefore u = \frac{dr}{dt}$$

Also, $v = \lim_{\delta t \to 0} \frac{\text{displacement in direction PM in time } \delta t}{\delta t}$

$$= \lim_{\delta t \to 0} \frac{MQ}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{(r+\delta r)\sin\delta\theta}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{(r+\delta r)\left\{\delta\theta - \frac{(\delta\theta)^3}{\lfloor 3} + \cdots\right\}}{\delta t}$$

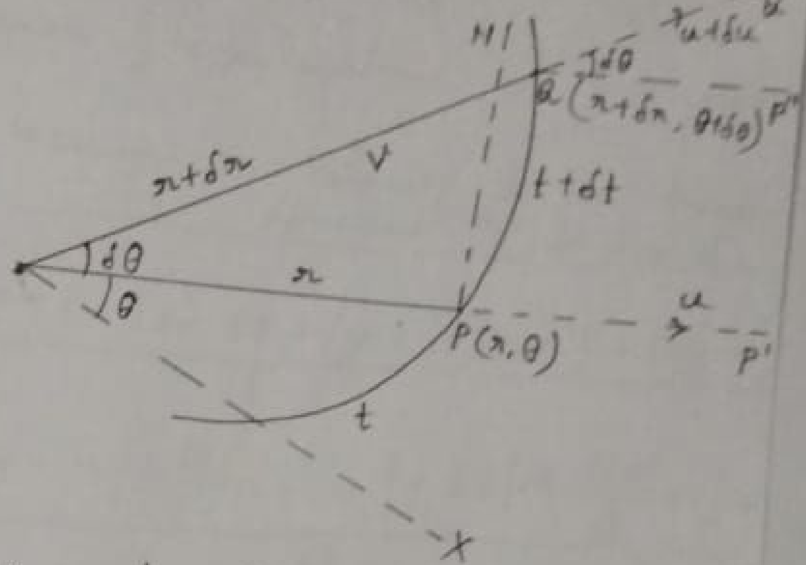
$$= \lim_{\delta t \to 0} \left[\frac{r\delta\theta}{\delta t} + \frac{\delta r}{\delta t}\left\{\delta\theta - \frac{(\delta\theta)^3}{\lfloor 3} + \cdots - r\frac{(\delta\theta)^3}{\delta t \lfloor 3} + \cdots\right]\right.$$

$$= \frac{r\,d\theta}{dt}$$

we now find expressions for the radial and transverse acceleration :-

Let at time t, where the particle is at $P(r, \theta)$, its radial and transverse velocities be u and v respectively.

Then at time $t+\delta t$, when the particle is at $Q(r+\delta r, \theta+\delta\theta)$, its radial and transverse velocities are $u+\delta u$, $v+\delta v$.



With reference to the above figure, let OP'' be drawn parallel to OP.
Then $\angle P''QQ' = \delta\theta$

Now acceleration is the time rate of increase in velocity.
$\therefore$ Radial acceleration at P is given by

$$R = \lim_{\delta t \to 0} \frac{\text{increase in velocity in the direction OP in limit } \delta t}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{\left\{(u+\delta u)\cos\delta\theta + (v+\delta v)\cos\left(\frac{\pi}{2}+\delta\theta\right)\right\} - u}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{u+\delta u\left[1-\frac{(\delta\theta)^2}{2}+\dots\right]-(v+\delta v)\sin\delta\theta - u}{\delta t}$$

$$= \lim_{\delta t \to 0}\left[\frac{\left\{-u\frac{(\delta\theta)^2}{\lfloor 2}+\dots\right\}}{\delta t} + \frac{\delta u}{\delta t}\left\{1-\frac{(\delta\theta)^2}{\lfloor 2}+\dots\right\} - \frac{(v+\delta v)}{\delta t}\left\{\delta\theta-\frac{(\delta\theta)^3}{\lfloor 3}+\dots\right\}\right]$$

$$= \frac{du}{dt} - \frac{v\,d\theta}{dt}$$

$$= \frac{d}{dt}\left(\frac{dr}{dt}\right) - \left(r\frac{d\theta}{dt}\right)\frac{d\theta}{dt} - \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2$$

The transverse acceleration at P is given by

$$T = \lim_{\delta t \to 0} \frac{\text{increase in velocity in the transverse direction PM in time } t}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{\left\{(u+\delta u)\sin\delta\theta + (v+\delta v)\sin\left(\frac{\pi}{2}+\delta\theta\right)\right\} - v}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{u\delta\theta - u\left\{\frac{(\delta\theta)^3}{3} - \cdots\right\} + \delta u\left\{\delta\theta - \frac{(\delta\theta)^3}{\lfloor 3} + \cdots\right\} - v}{\delta t}$$

$$= u\frac{d\theta}{dt} + \frac{dv}{dt}$$

$$= \frac{dr}{dt}\frac{d\theta}{dt} + \frac{d}{dt}\left(r\frac{d\theta}{dt}\right)$$

$$= \frac{dr}{dt}\frac{d\theta}{dt} + \frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}$$

$$= 2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}$$

This can also be written as

$$T = \frac{1}{r}\left[r^2\frac{d^2\theta}{dt^2} + \frac{d\theta}{dt}\cdot\frac{d}{dt}(r^2)\right]$$

$$= \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)$$