Q.7) Motion in a plane under inverse square law.

A particle moves in a plane so that its acceleration is always directed to a fixed point in the plane and is equal to $\mu \div (\text{distance})^2$ show that its path is a conic section and distinguish the three cases that arise.

The $p-r$ differential equation of the path is

$$\frac{h^2}{p^3}\frac{dp}{dr} = \frac{\mu}{r^2}$$

Integration gives

$$\frac{-h^2}{2p^2} = \frac{-\mu}{r} + \text{Constant}$$

or, $$\frac{h^2}{p^2} = \frac{2\mu}{r} + c \quad \text{—— (1)}$$

where C is a Constant.

Now whatever be the Constant C, the equation (1) is the (p, r) equation of a Conic section where focus is at the pole.

Hence the path of the particle is a Conic section whose focus is at the Centre of force.

Second part :-

Three cases arise according as C is negative, zero or positive.

Case I Let c be negative

Comparing ① with $\frac{b^2}{p^2} = \frac{2a}{r} - 1$

which represents an ellipse with 2a and 2b as the transverse and the conjugate axes, we see that in this case the path is an ellipse.

Since ① may be written as

$$\frac{\frac{h^2}{(-c)}}{p^2} = \frac{2\left(\frac{\mu}{-c}\right)}{r} - 1$$

We see that the transverse and the conjugate axes of the elliptical path are respectively $\frac{-2\mu}{c}$ and $2h\sqrt{-\frac{1}{c}}$

Case II Let $C = 0$

In this case ① becomes

$$\frac{h^2}{p^2} = \frac{2\mu}{r}$$

$$\text{or, } p^2 = \frac{h^2}{2\mu} r$$

which represents a parabola. Hence in, this case the path is a parabola whose latus rectum is $\frac{h^2}{2\mu}$

Case III Let c be positive

Comparing (1) with

$$\frac{b^2}{p^2} = \frac{2a}{r} + 1$$

which represents a hyperbola with 2a and 2b as the transverse and the Conjugate axes, we see that in this case the path is a hyperbola whose transverse and Conjugate axes are respectively $\frac{2\mu}{c}$ and $2h\sqrt{\frac{1}{c}}$

Now, if v be the velocity at any point of the path we have

$$v^2 = \frac{h^2}{p^2}$$

* ∴ From (1) we have $v^2 = \frac{2\mu}{r} + c$ —— (2)

Hence it follows that the path is an ellipse a parabola or a hyperbola according as $v^2 \lesseqgtr \frac{2\mu}{r}$

But the square of the velocity that would be acquired in falling from infinity to the point r.

$$= 2\int_{\infty}^{r}\left(-\frac{\mu}{r^2}\right)dr$$

$$= \frac{2\mu}{r}$$

Hence finally we arrive at the following result :-

The path is an ellipse, a parabola or a hyperbola according as the velocity at any point of the path is less than, equal to or greater than that acquired in falling from infinity to that point.