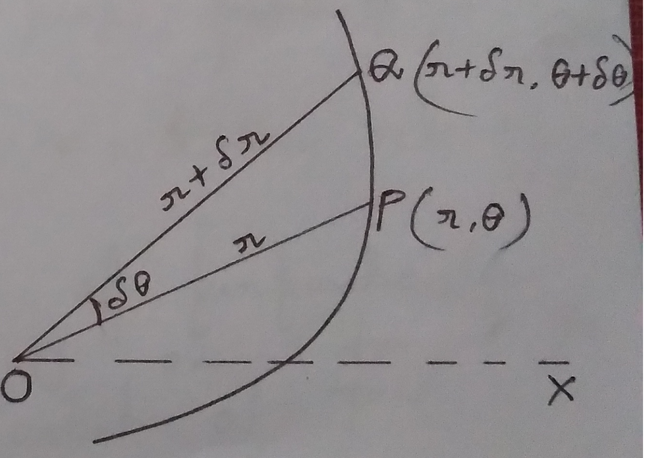
Q. 5 In every central orbit, the sectorial area traced out by the radius vector to the centre of force increases uniformly per unit time, and the linear velocity varies inversely as the perpendicular from the centre upon the tangent to the path.

Proof :-
Let O be the centre of force. With O as pole let $P(r, \theta)$ and $Q(r+\delta r, \theta+\delta\theta)$ be the position of the moving particle at time t and $t+\delta t$ respectively.



Now the area of the sector POQ,

$= \frac{1}{2} r(r+\delta r) \sin \delta\theta$

∴ Rate of description of sectorial area

$$= \lim_{\delta t \to 0} \frac{\frac{1}{2} r(r+\delta r) \sin\delta\theta}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{\frac{1}{2}(r^2 + r\delta r)}{\delta t} \left\{ \delta\theta - \frac{(\delta\theta)^3}{\lfloor 3} + \cdots \right\}$$

since $\delta\theta \to 0$ and $\delta r \to 0$ as $\delta t \to 0$ the above limit $= \frac{1}{2} r^2 \frac{d\theta}{dt}$ ——— (1)

But since there is no transverse acceleration in a Central orbit, we have

$$\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)=0$$

or, $r^2\dfrac{d\theta}{dt} = \text{Constant} = h$ (say)

Hence from ① rate of description of sectorial area $= \frac{1}{2}h = \text{const}$.

2nd part

The rate of description of sectional is also = Const.

$$= \lim_{\delta t\to 0} \frac{1}{2}\cdot PQ\,(\text{perp. from } O \text{ on } PQ)$$

$$= \lim_{\delta t\to 0} \left(\frac{1}{2}\frac{\delta s}{\delta t}\times \text{perp. from } O \text{ on } PQ\right)$$

$= \frac{1}{2}vp$ where v = linear velocity at P.

and p = length of perpendicular from O on tangent to the orbit at P.

$\therefore$ From ①,

$$\boxed{vp = r^2\frac{d\theta}{dt} = h}$$

$$\therefore\ v = \frac{h}{p} \propto \frac{1}{p}.$$

This proves the second part.