PG. IV sem. Paper - 1(I) plasma physics. Unit - 3. Dr. Kumar Balwant Singh. Physics:

Waves in Unmagnetized Plasma

To study the wave in unmagnetized plasma, we set **B**=**H**=0. As a result, the Maxwell's equations for the unmagnetized plasma can be given as:

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_0} \quad \text{------- (1)}$$

$$\nabla \cdot \boldsymbol{B} = 0 \quad \text{------- (2)}$$

$$\nabla \times \boldsymbol{E} = 0 \quad \text{------- (3)}$$

$$\boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t} = 0 \quad \text{------- (4)}$$

From equation (3), we can write

$$\boldsymbol{E} = -\nabla \phi \quad \text{-------- (5)}$$

The electric field **E** serves as a perturbation in plasma to induce waves.

Then from equation (1), we get:

$$\nabla^2 \phi = -\frac{\rho}{\varepsilon_0} \quad \text{-------- (6)}$$

Note that due to the perturbing electric field, there will be a perturbation in the number density of charges at a given instant of time in the plasma. Lets assume that, in an equilibrium state, the number density of positive ions and electrons are n_0 because of the quasi-neutrality condition. And after the introduction perturbing electric field in the plasma, the number density of ions (n_i) and number density of electrons (n_e) are:

$$n_i = n_0 + n_{1i} \quad \text{--------- (7)}$$

$$n_e = n_0 + n_{1e} \quad \text{--------- (8)}$$

Here n_{1i} and n_{1e} represent the change in equilibrium densities of ions and electrons (n_0) due to the perturbation. Further, we consider the perturbation to be small such that $|n_{1i}| << n_0$; $|n_{1e}| << n_0$.

As a result, the charge density

$$\rho = n_i q_i + n_e q_e$$
$$= (n_0 + n_{1i})\, e - (n_0 + n_{1e})\, e = (n_{1i} - n_{1e})\, e$$

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Now equation (6) can be written as:

$$\nabla^2 \phi = \frac{(n_{1e} - n_{1i})e}{\varepsilon_0} \qquad \text{---------} \ (9)$$

For a time being, here we make an assumption that perturbation is so small that the heavier ions don't respond to the perturbation and only electrons get affected. As a result, n_{1i} is approximately zero. So, under this assumption, oscillation in electron density leads to wave, while ions serve as a constant background to the wave. Because the assumption, the equation (9) modifies as:

$$\nabla^2 \phi = \frac{n_{1e} e}{\varepsilon_0} \qquad \text{---------} \ (10)$$

Now to study such a wave, we only need to consider the equation of motion of electron fluid, while ion fluid can be considered in a stationary state. The equation of motion of electron fluid is then

$$n_e m_e \left(\frac{\partial \mathbf{u}_e}{\partial t} + \mathbf{u}_e \cdot \nabla \mathbf{u}_e\right) = -\nabla p_e - n_e e \mathbf{E} \qquad \text{---------} \ (11)$$

where the magnetic field **B** =0 in the Lorentz force term for the unmagnetized plasma. For simplicity, if we consider the plasma to be in isothermal state then,

$$p_e = n_e T_e \qquad \text{---------} \ (12)$$

where we set the Boltzmann constant $k_B = 1$. Then equation (11) becomes:

$$n_e m_e \left(\frac{\partial \mathbf{u}_e}{\partial t} + \mathbf{u}_e \cdot \nabla \mathbf{u}_e\right) = -\nabla (n_e T_e) - n_e e \mathbf{E}$$

$$m_e \left(\frac{\partial \mathbf{u}_e}{\partial t} + \mathbf{u}_e \cdot \nabla \mathbf{u}_e\right) = -\frac{1}{n_e} \nabla (n_e T_e) - e \mathbf{E} \qquad \text{---------} \ (13)$$

The mass continuity equation for electron fluid is

$$\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e \mathbf{u}_e) = 0 \qquad \text{----------} \ (14)$$

Note that, due to an introduction of perturbing electric field **E**, there will be a change in the electron fluid velocity $\mathbf{u}_e$ according to equation (13). Then from equation (14), the changed velocity modifies the number density of electron n_e. This change in density modifies the electric field via equation (10) which once again modifies $\mathbf{u}_e$ and

$\mathbf{n}_e$ from equations (13) and (14). This feedback mechanism continues and wave is produced in the plasma.

To study the properties of the wave, we need to obtain the corresponding dispersion relation. For the purpose, we need to get linearized equations. For this, we first consider an equilibrium state of the plasma which is characterized by

velocity $\mathbf{u}_{0e} = 0$,
number density $n_{0e} = n_0$
Electric field $\mathbf{E}_0 = 0$

Now, after the introduction of perturbing electric field $\boldsymbol{E} = -\nabla\phi$, the plasma gets perturbed from the equilibrium state and the perturbed state is characterized by

velocity $\mathbf{u}_e = \mathbf{u}_{0e} + \mathbf{u}_{1e} = \mathbf{u}_{1e}$
number density $n_e = n_{0e} + n_{1e} = n_0 + n_{1e}$
Electric field $\boldsymbol{E} = -\nabla\phi$

Then, for the perturbed state, the momentum transport equation (13) becomes:

$$m_e\left(\frac{\partial \boldsymbol{u}_{1e}}{\partial t} + \boldsymbol{u}_{1e}\cdot\nabla \boldsymbol{u}_{1e}\right) = -\frac{1}{(n_0+n_{1e})}\nabla((n_0+n_{1e})T_e) + e\nabla\phi$$

Neglecting the multiplication of perturbing quantities, we get,

$$m_e\frac{\partial \boldsymbol{u}_{1e}}{\partial t} = -\frac{T_e}{(n_0+n_{1e})}\nabla n_{1e} + e\nabla\phi$$

$$m_e\frac{\partial \boldsymbol{u}_{1e}}{\partial t} = -\frac{T_e}{n_0}\nabla n_{1e} + e\nabla\phi$$

$$\frac{\partial \boldsymbol{u}_{1e}}{\partial t} = -\frac{T_e}{n_0 m_e}\nabla n_{1e} + \frac{e}{m_e}\nabla\phi \qquad \text{---------} \ (15)$$

Equation (15) is the linearized momentum transport equation.

Now, the mass continuity equation (14) for the perturbed state is

$$\frac{\partial(n_0+n_{1e})}{\partial t} + \nabla\cdot((n_0+n_{1e})\boldsymbol{u}_{1e}) = 0$$

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Neglecting the terms having two perturbing quantities, we get the linearized mass continuity equation:

$$\frac{\partial n_{1e}}{\partial t} + n_0 \nabla \cdot (\boldsymbol{u}_{1e}) = 0 \qquad \text{---------- (16)}$$

Equations (10), (15) and (16) represent the governing equations for the variations in the perturbing variables Φ, $\mathbf{u}_{1e}$, n_{1e}. Now to obtain the dispersion relation, we convert these equations into algebric equations by considering the perturbing variables in plane wave forms, i.e.

$$\phi \rightarrow \phi_0 \exp i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)$$

$$\boldsymbol{u}_{1e} \rightarrow \boldsymbol{u}_{1e0} \exp i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)$$

$$n_{1e} \rightarrow n_{1e0} \exp i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)$$

As a result of the forms, time derivative and spatial derivative operators in equations (10), (15) and (16) can be replaced as:

$$\frac{\partial}{\partial t} \rightarrow -i\omega$$

$$\nabla \rightarrow i\boldsymbol{k}$$

With the replacements of these operators, equation (15) modifies as:

$$\boldsymbol{u}_{1e} = \frac{T_e}{n_0 m_e} n_{1e} \boldsymbol{k} - \frac{e\phi}{m_e \omega} \boldsymbol{k}$$

$$\boldsymbol{u}_{1e} = \frac{e\phi}{m_e \omega} \boldsymbol{k} - \frac{v_{th}^2}{n_0} n_{1e} \boldsymbol{k} \qquad \text{---------- (17)}$$

where the thermal speed is $v_{th}^2 = T_e/m_e$.

Now, with the replacement of the operators in equation (16), we can get:

$$n_{1e} = \frac{n_0}{\omega} \boldsymbol{k} \cdot \boldsymbol{u}_{1e} \qquad \text{---------- (18)}$$

Put expression of $\mathbf{u}_{1e}$ from equation (17) to equation (18):

$$n_{1e} = \frac{n_0}{\omega}\mathbf{k}\cdot\left(\frac{e\phi}{m_e\omega}\mathbf{k} - \frac{v_{th}^2}{n_0} n_{1e}\mathbf{k}\right)$$

Further simplifications provide us:

$$n_{1e} = -\frac{n_0 e k^2 \phi}{m_e(\omega^2 - k^2 v_{th}^2)}$$

$$n_{1e} = -\frac{n_0 e^2 k^2 \phi}{m_e \varepsilon_0 (\omega^2 - k^2 v_{th}^2)} \frac{\varepsilon_0}{e}$$

$$n_{1e} = -\frac{\omega_p^2}{(\omega^2 - k^2 v_{th}^2)} \frac{k^2 \varepsilon_0}{e}\phi$$

where $\omega_p = \sqrt{\frac{n_0 e^2}{m_e \varepsilon_0}}$ represents the plasma oscillation frequency. Furthermore,

$$n_{1e} = \chi_e \frac{k^2 \varepsilon_0}{e}\phi \qquad \text{---------- (19)}$$

where $\chi_e = -\frac{\omega_p^2}{(\omega^2 - k^2 v_{th}^2)}$ is known as electron plasma susceptibility.

Ultimately, we use the replacement of operators in equation (10) and get:

$$-k^2\phi = \frac{e}{\varepsilon_0} n_{1e}$$

$$\phi = -\frac{e}{\varepsilon_0 k^2} n_{1e} \qquad \text{---------- (20)}$$

Use equation (19) into equation (20), we can obtain

$$1 + \chi_e = 0$$

$$1 - \frac{\omega_p^2}{\omega^2 - k^2 v_{th}^2} = 0$$

$$\omega^2=\omega_p^2+k^2v_{th}^2 \qquad \text{-----------} (21)$$

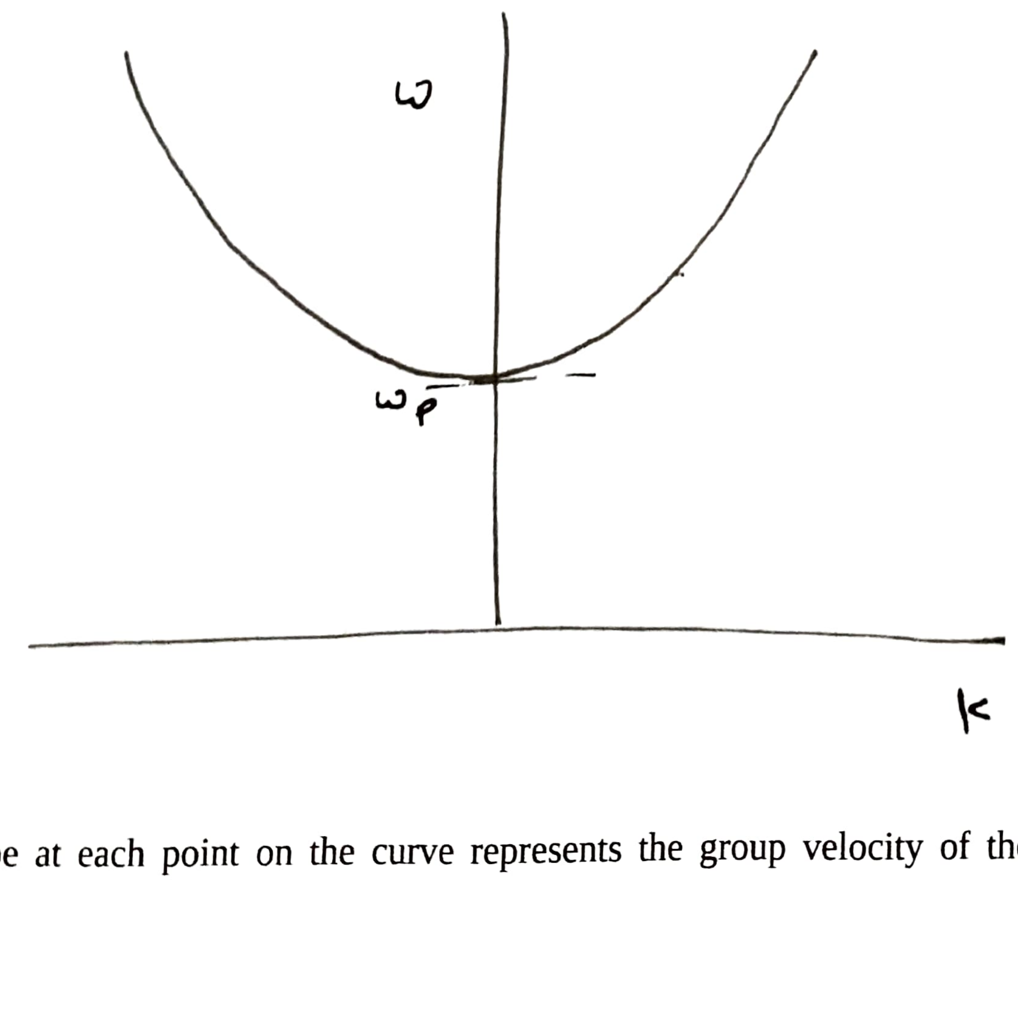
This is called the dispersion relation for the waves in unmagnetized plasma when ions are considered to be stationary under perturbation.

Note that when temperature T_e =0 i.e. $v_{th}=0$ then $\omega=\omega_p$. This implies that there is only plasma oscillation and group velocity $v_g = d\omega/dk = 0$. In this situation, there is no propagation of energy and, therefore, waves are absent. This indicates that it is the finite temperture of plasma which is responsible for the generation of wave motion. Such waves are called *"plasma wave"* or *"Langmuir wave"*. With finite temperature thermal speed is not zero which provides the finite value of group velocity.

From equation (21) we can write

$$\omega^2-\omega_p^2=k^2v_{th}^2$$

Since $k^2v_{th}^2>0$, therefore $\omega>\omega_p$. Hence angular frequency of plasma wave is always greater than plasma oscillation frequency. The plot of the dispersion relation (21) is illustrated as:



The slope at each point on the curve represents the group velocity of the plasma wave.