Ion Acoustic Waves

In the previous lecture on "Waves in Unmagnetized Plasma", we have studied "plasma waves" in which we neglected the ion's motion. For this, we considered the following Poisson's equation for perturbing electric potential,

$$\nabla^2 \phi = \frac{n_{1e} e}{\varepsilon_0}$$

But, if we consider the low frequency perturbation such that the frequency associated with the perturbation $\omega << kv_{th}$, then the motion of ions can not be neglected. For such a low frequency perturbation, change in number density of ions due to perturbation is finite and, as a result, the above Poisson's equation modifies as:

$$\nabla^2 \phi = \frac{(n_{1e} - n_{1i}) e}{\varepsilon_0} \qquad \text{---------- (1)}$$

For the completeness, we would like to mention that due to the perturbing electric field, there will be a perturbation in the number density of charges at a given instant of time in the plasma. If, in an equilibrium state, the number density of positive ions and electrons are n_0 because of the quasi-neutrality condition. And after the introduction perturbing electric field in the plasma, the number density of ions (n_i) and number density of electrons (n_e) become $n_i = n_0 + n_{1i}$ and $n_e = n_0 + n_{1e}$. Here n_{1i} and n_{1e} represent the change in equilibrium densities of ions and electrons (n_0) due to the perturbation. Further, we consider the perturbation to be small such that $|n_{1i}| << n_0$; $|n_{1e}| << n_0$.

Furthermore from calculations related to electron fluid done in the previous lecture, we know that

$$n_{1e} = \chi_e \frac{k^2 \varepsilon_0}{e} \phi \qquad \text{---------- (2)}$$

where $\chi_e = -\frac{\omega_p^2}{(\omega^2 - k^2 v_{th}^2)}$ is known as electron plasma susceptibility.

In a similar way to the electron fluid case, to include effects of the motion of ion fluid, we consider an equilibrium state and then perturbed it by an electric field perturbation. The perturbed state of ion fluid is characterized by

Ion fluid velocity $\mathbf{u_i} = \mathbf{u_{0i}} + \mathbf{u_{1i}} = \mathbf{u_{1i}}$

Ion number density $n_i = n_{0i} + n_{1i} = n_0 + n_{1i}$

Perturbing Electric field $\boldsymbol{E} = -\nabla \phi$

Pressure associated with ion's motion $P_i = n_i T_i$

Then, the linearized momentum transport equation for ion fluid is:

$$m_i\left(\frac{\partial \boldsymbol{u}_{1i}}{\partial t} + \boldsymbol{u}_{1i} \cdot \nabla \boldsymbol{u}_{1i}\right) = -\frac{1}{(n_0 + n_{1i})} \nabla\left((n_0 + n_{1i}) T_i\right) - e \nabla \phi$$

$$\frac{\partial \boldsymbol{u}_{1i}}{\partial t} = -\frac{T_i}{n_0 m_i} \nabla n_{1i} - \frac{e}{m_i} \nabla \phi \qquad \text{----------- (3)}$$

Linearized mass continuity equation for ion's fluid is:

$$\frac{\partial n_{1i}}{\partial t} + n_0 \nabla \cdot (\boldsymbol{u}_{1i}) = 0 \qquad \text{----------- (4)}$$

Equations (2), (3) and (4) represent the governing equations for the variations in the perturbing variables Φ, $\mathbf{u}_{1i}$, n_{1i}. Now to obtain the dispersion relation, we convert these equations into algebric equations by considering the perturbing variables in plane wave forms, i.e.

$$\phi \rightarrow \phi_0 \exp i(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)$$

$$\boldsymbol{u}_{1i} \rightarrow \boldsymbol{u}_{1i0} \exp i(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)$$

$$n_{1i} \rightarrow n_{1i0} \exp i(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)$$

As a result of the forms, time derivative and spatial derivative operators in equations (2), (3) and (4) can be replaced as:

$$\frac{\partial}{\partial t} \rightarrow -i\omega$$

$$\nabla \rightarrow i\boldsymbol{k}$$

Using this replacement of operators and, then, following steps of the previous lecture, for the ion fluid also we can get:

$$n_{1i} = \frac{n_0 e k^2 \phi}{m_i(\omega^2 - k^2 v_{thi}^{\ 2})}$$

where $v_{thi} = T_i/m_i$ is the thermal speed of ions.

$$n_{1i} = \frac{n_0 e^2 k^2 \phi}{m_i \varepsilon_0 \left(\omega^2 - k^2 v_{thi}^2\right)} \frac{\varepsilon_0}{e}$$

$$n_{1i} = \frac{\omega_{pi}^2}{\left(\omega^2 - k^2 v_{thi}^2\right)} \frac{k^2 \varepsilon_0}{e} \phi$$

where $\omega_{pi} = \sqrt{\frac{n_0 e^2}{m_i \varepsilon_0}}$ represents the ion plasma oscillation frequency. Furthermore,

$$n_{1i} = -\chi_i \frac{k^2 \varepsilon_0}{e} \phi \qquad \text{---------- (5)}$$

where $\chi_i = -\frac{\omega_{pi}^2}{\left(\omega^2 - k^2 v_{thi}^2\right)}$ is known as ion plasma susceptibility.

Now, we use the replacement of operators in equation (10) and get:

$$-k^2 \phi = \frac{e}{\varepsilon_0} \left(n_{1e} - n_{1i}\right)$$

$$\phi = -\frac{e}{\varepsilon_0 k^2} \left(n_{1e} - n_{1i}\right) \qquad \text{----------- (6)}$$

Use equation (19) into equation (20), we can obtain

$$1 + \chi_e + \chi_i = 0$$

$$1 - \frac{\omega_p^2}{\left(\omega^2 - k^2 v_{th}^2\right)} - \frac{\omega_{pi}^2}{\left(\omega^2 - k^2 v_{thi}^2\right)} = 0$$

Since for low frequency perturbation $\omega << kv_{th}$, the above equation modifies as:

$$1 + \frac{\omega_p^2}{k^2 v_{th}^2} - \frac{\omega_{pi}^2}{\left(\omega^2 - k^2 v_{thi}^2\right)} = 0$$

$$1 + \frac{\omega_p^2}{k^2 v_{th}^2} = \frac{\omega_{pi}^2}{\left(\omega^2 - k^2 v_{thi}^2\right)}$$

$$\omega^2 - k^2 v_{thi}^2 = \frac{\omega_{pi}^2}{1 + \frac{\omega_p^2}{k^2 v_{th}^2}}$$

$$\omega^2 = \frac{\omega_{pi}^2}{1 + \frac{\omega_p^2}{k^2 v_{th}^2}} + k^2 v_{thi}^2 \qquad \text{------------ (7)}$$

This is called the dispersion relation for Ion Acoustic wave.