# W.K.B. Approximation.

The approximate treatment of the Schrödinger wave eqn. which shows its connection with the quantization rules. It is expansion of the wave function in powers of $\hbar$, which, although of a semiconvergent or asymptotic character, is nevertheless also useful for the approximate solution of quantum mechanical problems in approximate cases. This method is called the Wenzel-Kramers-Brillouin or WKB approximation;

A solution $\psi(r, t)$ of the Schrödinger wave eqn. is

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2\mu} \nabla^2 \psi + V(r)\psi \quad \text{—— (1)}$$

Can be written in the form

$$\psi(r, t) = A \exp \frac{iW(r, t)}{\hbar}$$

in which case W satisfies the eqn.

$$\frac{\partial W}{\partial t} + \frac{1}{2\mu}(\nabla W)^2 + V - \frac{i\hbar}{2\mu}\nabla^2 W = 0 \quad \text{—— (2)}$$

In the classical limit ($\hbar \to 0$), Eqn. (2) is the same as Hamilton's partial differential eqn. for the principal function W;

$$\frac{\partial W}{\partial t} + H(r, p) = 0, \quad P = \nabla W$$

Since the momentum of the particle is the gradient of $W$, the possible trajectories are orthogonal to the surfaces of constant $W$ and hence, in the classical limit, to the surfaces of constant phase of the wave function $\psi$.

Thus in this limit the rays associated with $\psi$ are the possible paths of the classical particle.

If $\psi$ is an energy eigenfunction $u(r)e^{-iEt/\hbar}$, $W$ can be written

$$W(r,t) = S(r) - Et$$

In this case, we have that,

$$u(r) = A \exp \frac{iS(r)}{\hbar}$$

$$\frac{1}{2\mu}(\nabla S)^2 - [E - V(r)] - \frac{i\hbar}{2\mu}\nabla^2 S = 0 \qquad (3)$$

The WKB method obtains the first two terms of an expansion of $S$ in powers of $\hbar$, in the one dimensional case.

<u>Approximate Solutions:-</u>

The basic eqn that we consider is written in one of the forms

$$\frac{d^2u}{dx^2} + k^2(x)\, u = 0 \qquad k^2 > 0 \qquad (4)$$

$$\frac{d^2u}{dx^2} - k^2(x)\,u = 0 \qquad k^2 > 0 \quad \text{——— (5)}$$

so that K and k are always real. These are equivalent to the one-dimensional wave equation, if we put.

$$K(x) = +\frac{1}{\hbar}\{2\mu[E - V(x)]\}^{1/2} \quad \text{when } V(x) < E$$

$$k(x) = +\frac{1}{\hbar}\{2\mu[V(x) - E]\}^{1/2} \quad \text{when } V(x) > E$$

——— (6)

Eqn. (5) and (6) are also equivalent to the radial wave eqn. if $x$ is replaced by $r$, $V(r)$ is replaced by

$$V(r) + \frac{\hbar^2 l(l+1)}{2\mu r^2}$$

and $u$ is equal to $r$ times the radial wave function. We shall be able to generalize the resulting expression for $u(x)$ to obtain solutions of equation (4); we shall be able to generalize the resulting expression for $u(x)$ to obtain solutions of eqn. (4); we put,

$$u(x) = A\,e^{iS(x)/\hbar}$$