Bloch Theorem.

Statement: The solutions of the Schrodinger equation for a periodic potential must be of the form

$$\psi_K(r) = u_K(r)\exp(iK\cdot r), \quad \text{—— (1)}$$

Here, $u_K(r)$ has the period of the crystal lattice with $u_K(r) = u_K(r+T)$.
T is the crystal translation vector.

Explanation: The eigenfunctions of the wave equation for a periodic potential are the product of a plane wave exp $(iK\cdot r)$ times a function $u_K(r)$ with the periodicity of the crystal lattice. The function $\psi_K(r)$ in eqn (1) is called the Bloch function.

Proof:- We assume that ψ_K is nondegenerate. Consider N identical lattice points on a ring of length Na. The potential energy is periodic in a, with $U(x) = U(x+sa)$.
Here, s is an integer.

The ring has symmetry. We look for solutions of the wave equation such that

$$\psi(x+a) = C\psi(x), \quad \text{—— (2)}$$

Here, C is a constant. Then, on going once around the ring,

$$\psi(x+Na) = \psi(x) = C^N \psi(x),$$

because $\psi(x)$ must be single-valued.

So, C is one of the N roots of unity.

$$C = \exp(i 2\pi s/N); \quad s = 0, 1, 2, \ldots, N-1. \quad (3)$$

Suppose $U_K(x)$ has the periodicity a, so that $U_K(x) = U_K(x+a)$. Then

$$\psi(x) = U_K(x) \exp(i 2\pi s x / Na)$$

with $K = 2\pi s / Na$, satisfies eqn (2).

Thus the theorem is proved.

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