Division Ring

A ring R is said to be a division ring if it has unity element & each non-zero element of R possesses multiplicative inverse.

The division ring is also called a skew field or quasi field.

Ex. The set Q of rational numbers is a division ring.

Let $\frac{a}{b}, \frac{c}{d}, \frac{e}{f} \in Q$

We verify the postulates of division ring one by one

Closure law for addition

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd} \in Q$$

$\therefore$ closure law for addition holds

Associative law for addition

$$\left(\frac{a}{b} + \frac{c}{d}\right) + \frac{e}{f} = \frac{ad+bc}{bd} + \frac{e}{f} = \frac{adf+bcf+bde}{bdf}$$

Also, $\frac{a}{b} + \left(\frac{c}{d} + \frac{e}{f}\right) = \frac{a}{b} + \frac{cf+de}{df} = \frac{adf+bcf+bde}{bdf}$

$$\therefore \left(\frac{a}{b} + \frac{c}{d}\right) + \frac{e}{f} = \frac{a}{b} + \left(\frac{c}{d} + \frac{e}{f}\right)$$

i.e. associative law for addition holds

Existence of zero element

There exists a zero element $0 \in Q$ such that

$$\frac{a}{b} + 0 = 0 + \frac{a}{b} = \frac{a}{b}$$

Existence of additive inverse

For each element $\frac{a}{b} \in Q$ there exists an inverse $-\frac{a}{b} \in Q$ such that $\frac{a}{b} + \left(-\frac{a}{b}\right) = \left(-\frac{a}{b}\right) + \frac{a}{b} = 0$

Commutative law for addition

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd} = \frac{bc+ad}{bd} = \frac{cb+da}{db} = \frac{c}{d} + \frac{a}{b}$$

i.e. commutative law for addition holds.

Closure law for multiplication

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd} \in Q$$

i.e. closure law for multiplication holds

Associative law for multiplication

$$\left(\frac{a}{b}\cdot\frac{c}{d}\right)\cdot\frac{e}{f} = \frac{ac}{bd}\cdot\frac{e}{f} = \frac{ace}{bdf}$$

Also, $\frac{a}{b}\cdot\left(\frac{c}{d}\cdot\frac{e}{f}\right) = \frac{a}{b}\cdot\frac{ce}{df} = \frac{ace}{bdf}$

$$\therefore \left(\frac{a}{b}\cdot\frac{c}{d}\right)\cdot\frac{e}{f} = \frac{a}{b}\cdot\left(\frac{c}{d}\cdot\frac{e}{f}\right)$$

i.e. associative law for multiplication holds.

Existence of unity element

There exists an unity element $1 \in Q$ such that $\frac{a}{b}\cdot 1 = 1\cdot\frac{a}{b} = \frac{a}{b}$

Existence of inverse element (for multiplication)

For each non-zero element $\frac{a}{b} \in Q$ there exists a multiplicative inverse $\frac{b}{a} \in Q$ such that $\frac{a}{b}\cdot\frac{b}{a} = \frac{b}{a}\cdot\frac{a}{b} = 1$

Distributive laws

$$\frac{a}{b}\cdot\left(\frac{c}{d}+\frac{e}{f}\right) = \frac{a}{b}\cdot\frac{cf+de}{df} = \frac{acf+ade}{bdf}$$

Also, $\frac{a}{b}\cdot\frac{c}{d} + \frac{a}{b}\cdot\frac{e}{f} = \frac{ac}{bd} + \frac{ae}{bf} = \frac{acf+ade}{bdf}$

$$\therefore \frac{a}{b}\cdot\left(\frac{c}{d}+\frac{e}{f}\right) = \frac{a}{b}\cdot\frac{c}{d} + \frac{a}{b}\cdot\frac{e}{f}$$

Similarly we can show that

$$\left(\frac{c}{d}+\frac{e}{f}\right)\cdot\frac{a}{b} = \frac{c}{d}\cdot\frac{a}{b} + \frac{e}{f}\cdot\frac{a}{b}$$

i.e. distributive laws hold.

Thus Q is a division ring

Q. Prove that a division ring has no zero divisors

Let R be a division ring

Let $a, b \in R$ such that $ab = 0$

Let $a \neq 0$. Then a^{-1} exists.

Now $a^{-1}(ab) = a^{-1}0 = 0$

$\Rightarrow (a^{-1}a)b = 0$

$\Rightarrow eb = 0$

$\Rightarrow b = 0$. Hence $ab = 0 \Rightarrow b = 0$

Again let $b \neq 0$. Then b^{-1} exists.

Now $(ab)b^{-1} = 0b^{-1} = 0$

$\Rightarrow a(bb^{-1}) = 0$

$\Rightarrow ae = 0$

$\Rightarrow a = 0$. Hence $ab = 0 \Rightarrow a = 0$

Thus $ab = 0 \Rightarrow$ either $a = 0$ or $b = 0$. This shows that R has no zero divisors.