(10) Theorem: show that

(i) $[\vec{a}\ \vec{b}\ \vec{c}] = [\vec{b}\ \vec{c}\ \vec{a}] = [\vec{c}\ \vec{a}\ \vec{b}]$

i.e. a cyclic permutation of the three vectors $\vec{a}, \vec{b}, \vec{c}$ does not change the value of the scalar triple product.

(ii) $-[\vec{a}\ \vec{b}\ \vec{c}] = [\vec{b}\ \vec{a}\ \vec{c}] = [\vec{c}\ \vec{b}\ \vec{a}] = [\vec{a}\ \vec{c}\ \vec{b}]$

(i.e. every change of the cyclic order corresponds a negative sign in its value.)

(iii) $[\vec{a}\ \vec{a}\ \vec{c}] = [\vec{a}\ \vec{b}\ \vec{b}] = [\vec{a}\ \vec{b}\ \vec{a}] = 0$

(i.e. the scalar triple product $[\vec{a}\ \vec{b}\ \vec{c}] = 0$ if any two of the vectors are identical)

(iv) $\vec{a}\cdot(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\cdot\vec{c}$

(i.e. the positions of dot and cross can be interchanged without changing its value and sign.

Proof:- To prove (i):

we have $[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ —— (1)

since the interchange of any two rows of a determinant changes its sign we have

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = -\begin{vmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$$

$= [\vec{b}\ \vec{c}\ \vec{a}]$ using (1) —— (2)

∴ ~~From~~ Also ~~1, 2, 3 we have~~
~~$[\vec{a}, \vec{b}, \vec{c}]$~~

Also,

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = - \begin{vmatrix} c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$= [\vec{c}\ \vec{a}\ \vec{b}]$ using (1) —— (3)

From (1), (2), (3) we have

$$[\vec{a}\ \vec{b}\ \vec{c}] = [\vec{b}\ \vec{c}\ \vec{a}] = [\vec{c}\ \vec{a}\ \vec{b}]$$

Proved

Next, the proof of (ii) is similar to (i).

To prove (iii):

we have

$$[\vec{a}\ \vec{a}\ \vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0 \text{ (two rows are identical)}$$

To prove (iv): we have

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = [\vec{a}\ \vec{b}\ \vec{c}] = [\vec{c}\ \vec{a}\ \vec{b}] \text{ using (1)}$$

$$\vec{c} \cdot (\vec{a} \times \vec{b}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$$

Proved

Notes:- (a) The properties (i) and (ii) are Known as the parallelopiped laws

(b) The properties (i) and (ii) can also be proved from the geometrical meaning of the scalar triple product $[\vec{a}\ \vec{b}\ \vec{c}]$ as the volume of a parallelopiped having concurrent edges $\vec{a}, \vec{b}, \vec{c}$.

Distributive law for the cross product of vectors on using the scalar triple Product.

Theorem :- show that $\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b}+\vec{a}\times\vec{c}$ for any three vectors $\vec{a}, \vec{b}, \vec{c}$.

Proof:- We define a vector $\vec{d}$ by

$$\vec{d} = \vec{a}\times(\vec{b}+\vec{c}) - (\vec{a}\times\vec{b}) - (\vec{a}\times\vec{c}) \quad \text{--- (1)}$$

let $\vec{r}$ be any arbitrary non-zero vector.

$$\vec{r}\cdot\vec{d} = [\vec{r}\cdot[\vec{a}\times(\vec{b}+\vec{c}) - (\vec{a}\times\vec{b}) - (\vec{a}\times\vec{c})]]$$

$$= \vec{r}\cdot\{\vec{a}\times(\vec{b}+\vec{c})\} - \vec{r}\cdot(\vec{a}\times\vec{b}) - \vec{r}\cdot(\vec{a}\times\vec{c})$$

($\because$ using distributive law of scalar product of vectors)

$$= (\vec{r}\times\vec{a})\cdot(\vec{b}+\vec{c}) - (\vec{r}\times\vec{a})\cdot\vec{b} - (\vec{r}\times\vec{a})\cdot\vec{c}$$

($\because$ Position of dot and cross can be interchanged in the scalar triple product.

$$= (\vec{r}\times\vec{a})\cdot[(\vec{b}+\vec{c}) - \vec{b} - \vec{c}]$$

($\because$ Using distributive law of scalar product of vectors.

$$= (\vec{r}\times\vec{a})\cdot 0 = 0$$

$\therefore$ $\vec{r}\cdot\vec{d} = 0$ for all vectors $\vec{r}$

This emplies that $\vec{d} = 0$

$\therefore$ $\vec{a}\times(\vec{b}+\vec{c}) - (\vec{a}\times\vec{b}) - (\vec{a}\times\vec{c}) = 0$ using (1)

i.e. $\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b}+\vec{a}\times\vec{c}$ Proved

Condition of Coplanarity of three vectors $\vec{a}, \vec{b}, \vec{c}$

Theorem :- show that a necessary and sufficient condition for the three vectors $\vec{a}, \vec{b}, \vec{c}$ to be coplanar is that their scalar triple product $[\vec{a}\ \vec{b}\ \vec{c}]$ is zero.