Abstract Algebra

Direct sum of Submodules

Definition. If M is an R-module and if $M_1, M_2, \ldots, M_k$ are submodules of M then M is said to be the direct sum of $M_1, M_2, \ldots, M_k$ if every element of $m \in M$ can be uniquely expressed as

$$m = m_1 + m_2 + \cdots + m_k$$

where $m_1 \in M_1$, $m_2 \in M_2, \ldots, m_k \in M_k$.

It is usually denoted by $M = M_1 \oplus M_2 \oplus \cdots \oplus M_k$

Definition. If M is an R-module. Then a submodule M_1 of M is said to be a direct summand of M if there exists a submodule M_2 of M such that $M = M_1 \oplus M_2$

SUBMODULE GENERATED BY A SUBSET OF A MODULE

Let M be an R-module and S be any non-empty subset of M, then a submodule A of M is generated by S if $S \subseteq A$ and A is contained in every submodule of M containing S. Hence A is denoted by (S) i.e. $A = (S)$.

Notes :-

(i) The submodule of a unital R-module M generated by a subsets S of M consists of all linear combinations of elements of S.

CYCLIC MODULE

Definition 1 :- An R- module M is said to be cyclic if there is an element $m_0 \in M$ such that every $m \in M$ can be written as $m = rm_0$ for all $r \in R$. We also write $M = (m_0)$. Here m_0 is called generator of M.

Definition 2 :- An R- module M is said to be finitely generated if there exist elements $a_1, a_2, \ldots, a_n \in M$ such that every $m \in M$ may be written as
$$m = r_1 a_1 + r_2 a_2 + \cdots + r_n a_n \text{ for } r_i\text{'s} \in R$$

Notes :-

(i) If R be a Euclidean ring, then any finitely generated R- module M is the direct sum of a finite number of cyclic sub-modules.

Example 1 :- Prove that any unital irreducible R- module is cyclic.

Solution :- Let M be a unital irreducible R- module. Then we shall show that M is cyclic.

Since M is irreducible R- module so its only submodules are (0) and (M) itself.

If $M = (0)$, then trivially, M is cyclic.

Now, suppose $M \neq 0$, then there exists a non-zero element $m_0 \in M$.

Consider $A = \{rm_0 : r \in R\}$

clearly A is submodule of M and M is unital
$\therefore m_0 = 1 \cdot m_0 \in A \implies A \neq 0$

since M is irreducible, therefore $A = M$
Hence M is cyclic.

Example 2 Let R be a ring with unity. An R-module M is cyclic if and only if $M \cong R/I$ for some left ideal I of R.

Solution :- Let M be cyclic module generated by m_0 i.e. $M = \{rm_0 : r \in R\}$

Let $I = \{r \in R : rm_0 = 0\}$, then clearly I is a left ideal of R.

Define a mapping $f : R \to M$ given by

$$f(r) = rm_0, \quad r \in R$$

clearly f is an R-homomorphism and it is also onto.

Also, $\ker(f) = \{r \in R : rm_0 = 0\} = I$

Thus by fundamental theorem of R-homomorphism we have

$$R/I \cong M$$

Conversely, $M \cong R/I$. Since $I \in R$, then the left R-module R/I is generated by $1 + I \in R$ i.e. $R/I = (1 + I)$ so R/I is cyclic and $M \cong R/I$, hence M is cyclic.