* ## D'Alembert's Ratio Test :-

In positive term series $\sum a_n$ be as such that

$$\lim_{n \to \infty} \frac{a_n}{a_{n+1}} = \lambda \quad \text{--- (1)}$$

Then (i) $\sum a_n$ converges if $\lambda > 1$,

(ii) $\sum a_n$ diverges if $\lambda < 1$,

Test fails if $\lambda = 1$

or

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lambda$$

then (i) $\sum a_n$ converges for $\lambda < 1$

(ii) $\sum a_n$ diverges if $\lambda > 1$

Proof.

Case-I :- When $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lambda < 1$

By definition of limit, we can find a positive number $k (< 1)$ such that

$$\frac{a_{n+1}}{a_n} < k \quad \text{for all } n \geq m$$

leaving out the first m terms, let the series be $a_1 + a_2 + a_3 + \cdots \cdots$

So that $\frac{a_2}{a_1} < k$, $\frac{a_3}{a_2} < k$, $\frac{a_4}{a_3} < k$ ...... and so

Then
$$a_1 + a_2 + a_3 + \cdots\cdots + \infty$$
$$= a_1\left(1 + \frac{a_2}{a_1} + \frac{a_3}{a_2}\cdot\frac{a_2}{a_1} + \frac{a_4}{a_3}\cdot\frac{a_3}{a_2}\cdot\frac{a_2}{a_1}\cdots\cdots\right)$$
$$= a_1\left(1 + r + r^2 + r^3 + \cdots\cdots + \infty\right)$$
$$< a_1\left(\frac{1}{1-r}\right), \text{ which is finite quantity}$$
$$= a_1/1-r$$

Hence Σa_n is convergent for $\lambda < 1$

Case-II When $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lambda > 1$

By definition of limit, we can find m such that
$$\frac{a_{n+1}}{a_n} \geq m \geq 1 \text{ for all } n \geq m$$
$$\therefore \frac{a_{n+1}}{a_n} \geq 1 \quad \text{for } n \geq m$$

Leaving the first m terms, let the series be $a_1 + a_2 + a_3 + \cdots$

So that $\frac{a_2}{a_1} \geq 1$, $\frac{a_3}{a_2} \geq 1$, $\frac{a_4}{a_3} \geq 1$, ... and so on.

Then

$$a_1 + a_2 + a_3 + \cdots \text{———} + a_n$$

$$= a_1\left(1 + \frac{a_2}{a_1} + \frac{a_3}{a_2}\cdot\frac{a_2}{a_1} + \frac{a_4}{a_3}\cdot\frac{a_3}{a_2}\cdot\frac{a_2}{a_1} + \cdots\right)$$

$$\geq a_1(1 + 1 + 1\cdot 1 + 1\cdot 1\cdot 1 + \cdots\cdots)$$

$$\geq a_1(1 + 1 + 1 + 1 + \cdots \text{ to } n \text{ terms}) = a_1 n.$$

$$\because \lim_{n\to\infty} n a_1 \to \infty$$

Hence $\sum a_n$ is divergent for $\lambda > 1$

Ratio test fails when $\lambda = 1$

Consider, for example/instance the series

$$\sum a_n = \frac{1}{n^p}$$

$$\because \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lambda \Rightarrow \lim_{n\to\infty} \frac{n^p}{(n+1)^p}$$

$$\lim_{n\to\infty} \frac{n^p}{n^p(1+1/n)^p} = \lim_{n\to\infty} \frac{1}{(1+1/n)^p} = 1$$

Then for all p values $\lambda = 1$; whereas $\sum \frac{1}{n^p}$ Converges for $p > 1$ and diverges for $p < 1$

Hence $\lambda = 1$ both for convergence and divergence of $\sum a_n$

Exp. Test for convergence the seri..

$$\frac{1}{2}+\frac{1}{2\cdot 2^2}+\frac{1}{3\cdot 2^3}+\cdots$$

Sol. we have

$$a_n=\frac{1}{n\cdot 2^n}$$

$$\Rightarrow \quad a_{n+1}=\frac{1}{(n+1)\cdot 2^{n+1}}$$

$$\because \lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{n\cdot 2^n}{(n+1)2^{n+1}}$$

$$=\lim_{n\to\infty}\frac{1}{2(1+1/n)}$$

$$=(1/2)<1$$

Hence $\lambda<1$, by ratio test the given series $\sum\frac{1}{n2^n}$ is converg..

Que 1. Test for convergence the se..

$$\left(\frac{1}{2}\right)^2+\left(\frac{1}{3}\cdot\frac{2}{5}\right)^2+\left(\frac{1}{3}\cdot\frac{2}{5}\cdot\frac{3}{7}\right)^2+\cdots$$

$$\left(\because \text{Hint} \quad a_n=\left[\frac{1\cdot 2\cdot 3\cdots n}{3\cdot 5\cdot 7\cdots(2n+1)}\right]^2\right.$$

$$a_{n+1}=\left[\frac{1\cdot 2\cdot 3\cdots n(n+1)}{3\cdot 5\cdot 7\cdots(2n+1)(2n+3)}\right]^2$$