CONVEX SETS AND ITS PROPERTIES :-

Theorem VII:— If the objective function of a L.P.P. assumes its optimal value at more than one extreme point, then every Convex Combination of these extreme points gives the optimal value of the objective function.

Proof:— Let us Consider the L.P.P. as follows

$$\text{Max } Z = cx$$

$$\text{such that } Ax = b,\ x \geq 0.$$

Let $x_1, x_2, \ldots, x_k$ be the extreme points of the feasible region. If the objective function Z assumes its optimal value Z^* at the extreme points $x_1, x_2, \ldots, x_p$, $(p \leq k)$ then

$$Z^* = cx_1 = cx_2 = \cdots = cx_p.$$

If x_0 is the Convex Combination of the extreme points $x_1, x_2, \ldots, x_p$, then

$$x_0 = \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_p x_p,\ \lambda_i \geq 0\ \sum_{i=1}^{p} \lambda_i = 1$$

$$\therefore cx_0 = c\left[\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_p x_p\right]$$

$$= \lambda_1 cx_1 + \lambda_2 cx_2 + \cdots + \lambda_p cx_p$$

$$= \lambda_1 Z^* + \lambda_2 Z^* + \cdots + \lambda_p Z^*$$

$$= (\lambda_1 + \lambda_2 + \cdots + \lambda_p) Z^* = Z^*$$

$$\left[\because \sum_{i=1}^{n} \lambda_i = 1\right]$$

||

Hence the optimal value z^* is also attained at x_0 which is the convex combination of the extreme points at which optimal value occurs. Hence the theorem.

Examples :-

Ex:- 1 Find all the basic feasible solutions for the equations

$$2x_1 + 6x_2 + 2x_3 + x_4 = 3$$
$$6x_1 + 4x_2 + 4x_3 + 6x_4 = 2$$
$$x_i \geq 0$$

and determine the associated general convex combination of the extreme point solutions.

sol:- In matrix form the given system of equation can be written as

$$Ax = b$$

where $A = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$

$$\alpha_1 = \begin{bmatrix} 2 \\ 6 \end{bmatrix}, \alpha_2 = \begin{bmatrix} 6 \\ 4 \end{bmatrix}, \alpha_3 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \alpha_4 = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad b = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

This problem can have at most ${}^4C_3 = 6$ basic solutions. Now the six sets of two vectors are

$$B_1 = [\alpha_1, \alpha_2] = \begin{bmatrix} 2 & 6 \\ 6 & 4 \end{bmatrix}, B_2 = [\alpha_1, \alpha_3] = \begin{bmatrix} 2 & 2 \\ 6 & 4 \end{bmatrix}$$

$$B_3 = [\alpha_1, \alpha_4] = \begin{bmatrix} 2 & 1 \\ 6 & 6 \end{bmatrix}, \quad B_4 = [\alpha_2, \alpha_3] = \begin{bmatrix} 6 & 2 \\ 4 & 4 \end{bmatrix}$$

$$B_5 = [\alpha_2, \alpha_4] = \begin{bmatrix} 6 & 1 \\ 4 & 6 \end{bmatrix}, \quad B_6 = [\alpha_3, \alpha_4] = \begin{bmatrix} 2 & 1 \\ 4 & 6 \end{bmatrix}$$

Here

$$|B_1| = -28, \quad |B_2| = -4, \quad |B_3| = 6, \quad |B_4| = 16$$

$$|B_5| = 32, \quad |B_6| = 8$$

Since none of them is zero, therefore all these sets are L.I.

Hence all the six basic solutions exist.

If X_{B_i}, $i = 1, 2, \ldots 6$ are the two vectors of the basic variables associated to the sets B_i, $i = 1, 2, \ldots 6$ respectively. then

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = X_{B_1} = B_1^{-1} b = -\frac{1}{28} \begin{bmatrix} 4 & -6 \\ -6 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = X_{B_2} = B_2^{-1} b = -\frac{1}{4} \begin{bmatrix} 4 & -2 \\ -6 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 7/2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_4 \end{bmatrix} = X_{B_3} = B_3^{-1} b = \frac{1}{6} \begin{bmatrix} 6 & -1 \\ -6 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 8/3 \\ -7/3 \end{bmatrix}$$

$$\begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = X_{B_4} = B_4^{-1} b = \frac{1}{16} \begin{bmatrix} 4 & -2 \\ -4 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_2 \\ x_4 \end{bmatrix} = xB_5 = B_5^{-1} b = \frac{1}{32}\begin{bmatrix} 6 & -1 \\ -4 & 6 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = xB_6 = B_6^{-1} b = \frac{1}{8}\begin{bmatrix} 6 & -1 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Thus it is obvious that out of these only three basic solutions are B.F.S. (in which variables are non-negative). But the B.F.S's Correspond to the extreme points. Hence the only three extreme points solutions are given by

$$X_1 = \left(0, \frac{1}{2}, 0, 0\right), \quad X_2 = \left(0, \frac{1}{2}, 0, 0\right), \quad X_3 = \left(0, \frac{1}{2}, 0, 0\right)$$

Here $X_1 = X_2 = X_3$. Hence there is unique extreme point solution.