Linear programming

Theorem 1 :-

A hyperplane is a Convex set.

Proof Consider the hyperplane.

$$X = \{x : cx = z\}$$

Let x_1 and x_2 be any two points in the hyperplane X.

$\therefore cx_1 = z$ and $cx_2 = z$.

If $x_3 = \lambda x_1 + (1-\lambda) x_2$, $0 \leq \lambda \leq 1$

then $cx_3 = \lambda cx_1 + (1-\lambda) cx_2$

$= \lambda z + (1-\lambda) z = z$

which implies that

$x_3 = \lambda x_1 + (1-\lambda) x_2$ is also a point in the hyper-plane X.

Hence by definition, the hyper-plane X is a Convex set.

Theorem 2 :- Intersection of two Convex sets is also a Convex set.

Proof :- Consider two Convex sets X_1 and X_2. Let X_3 be the intersection of sets X_1 and X_2 i.e. $X_3 = X_1 \cap X_2$

If $x_1 \in X_1 \cap X_2 \Rightarrow x_1 \in X_1$ and $x_1 \in X_1$:

$0 \leq \lambda \leq 1$

and $x_2 \in X_1 \cap X_2 \Rightarrow x_2 \in X_1$ and $x_2 \in X_2$

Since X_1 and X_2 are Convex sets

$\therefore x_1 . x_2 \in X_1 \Rightarrow \lambda x_1 + (1-\lambda) x_3 \in X_1$:

$0 \leq \lambda \leq 1$

and $x_1, x_2 \in X_2 \Rightarrow \lambda x_1 + (1-\lambda)x_2 \in X_2,$
$0 \leq \lambda \leq 1$

Now, $\lambda x_1 + (1-\lambda)x_2 \in X_1$ and $\lambda x_1 + (1-\lambda)x_2 \in X_2$
$\Rightarrow \lambda x_1 + (1-\lambda)x_2 \in X_1 \cap X_2, \; 0 \leq \lambda \leq 1$

Hence by definition, $X_3 = X_1 \cap X_2$ is a Convex set.

Theorem 3:- The set of all convex combination of a finite number of points $x_1, x_2, \ldots, x_n$ is a Convex set.

Sol:- Let X be the set of all convex combinations of a finite number of points.

i.e. $X = \{x : x = \sum_{i=1}^{n} \lambda_i x_i, \; \sum_{i=1}^{n} \lambda_i = 1, \lambda_i \geq 0\}$

Let $u, v \in X$

$$u = \sum_{i=1}^{n} a_i x_i, \quad \sum_{i=1}^{n} a_i = 1, \quad a_i \geq 0$$

and $$v = \sum_{i=1}^{n} b_i x_i \quad \sum_{i=1}^{n} b_i = 1, \quad b_i \geq 0$$

Consider $w = \lambda u + (1-\lambda) v, \quad 0 \leq \lambda \leq 1$

$$\therefore w = \lambda \sum_{i=1}^{n} a_i x_i + (1-\lambda) \sum_{i=1}^{n} b_i x_i$$

$$= \sum_{i=1}^{n} \{\lambda a_i + (1-\lambda) b_i\} x_i$$

$$= \sum_{i=1}^{n} c_i x_i \quad \text{where } c_i = \lambda a_i + (1-\lambda) b_i$$

Now, $$\sum_{i=1}^{n} c_i = \sum_{i=1}^{n} \{\lambda a_i + (1-\lambda) b_i\}$$

$$= \lambda \sum_{i=1}^{n} a_i + (1-\lambda) \sum_{i=1}^{n} b_i$$

$$= \lambda \cdot 1 + (1-\lambda) \cdot 1 = 1 .$$

Also, $c_i = \lambda a_i + (1-\lambda) b_i \geq 0 \quad \forall i$

Hence $W = \sum_{i=1}^{n} c_i x_i$ is a Convex Combination of $x_1, x_2, \ldots, x_n$.

i.e. $w = \lambda u + (1-\lambda) v \in X, \quad 0 \leq \lambda \leq 1.$

Hence by definition X is a Convex set.