Linear Programming

Converse of theorem 5(V)

To prove that every extreme point of the convex set of feasible solutions is a B.F.S.

Let $X = [x_1, x_2, \ldots, x_n]$ be an extreme point. Now in order to prove that x is a B.F.S. we shall prove that the vectors associated with the positive elements of x are L.I.

Suppose that k-components as the first k components of x.

$$\therefore \sum_{i=1}^{k} x_i \alpha_i = b, \; x_i > 0, \; 1, 2, \ldots k \quad \text{——} \; (6)$$

where α_i is the column vector in A associated to the i^{th} variable in x.

If possible let the column vectors $\alpha_1, \alpha_2, \ldots, \alpha_k$ of matrix A be L.D. Then there exist some scalars $\lambda_i \; (i = 1, 2, \ldots, n)$, with at least one of them non-zero, s.t.

$$\sum_{i=1}^{k} \lambda_i \alpha_i = 0 \quad \text{——} \; (7)$$

From (6) and (7), for some arbitrary $\delta > 0$, we have

$$\sum_{i=1}^{k} x_i \alpha_i \pm \delta \sum_{i=1}^{k} \lambda_i \alpha_i = b$$

or

$$\sum_{i=1}^{k} (x_i \pm \delta\lambda_i) \alpha_i = b$$

from which it is obvious that the two points $(n-k)$ in number.

$$x_1^* = [x_1 + \delta\lambda_1, x_2 + \delta\lambda_2, \ldots, x_2 + \delta\lambda_2, 0, 0, \ldots 0]$$

$$x_2^* = [x_1 - \delta\lambda_1, x_2 - \delta\lambda_2, \ldots, x_2 - \delta\lambda_2, 0, 0, \ldots 0]$$

satisfy.

Also since $x_i > 0$, therefore taking δ such that

$$0 < \delta < \underset{i}{\text{Mini}} \left\{ \frac{x_i}{|x_i|} \right\}$$

we Conclude that first k Components of X_1^* and X_2^* are always positive. But the remaining Components of X_1^* and X_2^* are zero. which follows that X_1^* and X_2^* are feasible solutions from X.

Now

$$X_1^* + X_2^* = 2[x_1, x_2, \ldots, x_k, 0, 0, \ldots, 0]$$

$$\text{or, } \frac{1}{2}X_1^* + \frac{1}{2}X_2^* = [x_1, x_2, \ldots, x_2, 0, 0, \ldots, 0] = X$$

$$\text{or, } X = \lambda X_1^* + (1-\lambda) X_2^* \quad \text{where } \lambda = \frac{1}{2}$$

i.e. X can be expressed as a convex Combination of two distinct feasible solutions X_1^* and X_2^*. But this is a Contradiction as X is an extreme point. Hence the vectors $\alpha_1, \alpha_2, \ldots \alpha_k$ are L.I.

Further we Know that at most m vectors of E^m can be L.I. So $\alpha_1, \alpha_2, \ldots \alpha_k$ Cannot be more than m and hence the extreme point X will have atmost m non zero variables i.e. at least $(n-m)$ variables will be zero.

Thus X is a B.F.S. Hence every extreme point of the Convex set of feasible solutions in a B.F.S.