

Linear Programming

Converse of theorem 5(V)

To prove that every extreme point of the convex set of feasible solutions is a B.F.S.

Let $x = [x_1, x_2, \dots, x_n]$ be an extreme point. Now in order to prove that x is a B.F.S. we shall prove that the vectors associated with the positive elements of x are L.I.

Suppose that k components as the first k components of x .

$$\therefore \sum_{i=1}^k x_i \alpha_i = b, \quad x_i > 0, \quad i = 1, 2, \dots, k$$

where α_i is the i th column vector in A associated to the i th variable in x . (6)

if possible let the column vectors $\alpha_1, \alpha_2, \dots, \alpha_k$ of matrix A be L.D. Then there exist some scalars λ_i ($i = 1, 2, \dots, k$), with at least one of them non-zero, s.t.

$$\sum_{i=1}^k \lambda_i \alpha_i = 0 \quad \text{--- (7)}$$

From (6) and (7), for some arbitrary $\delta > 0$, we have

$$\sum_{i=1}^k x_i \alpha_i \pm \delta \sum_{i=1}^k \lambda_i \alpha_i = b$$

$$\text{or } \sum_{i=1}^k (x_i \pm \delta \lambda_i) \alpha_i = b$$

from which it is obvious that the two points $(n-k)$ in number.

$$x_1^* = [x_1 + \delta \lambda_1, x_2 + \delta \lambda_2, \dots, x_k + \delta \lambda_k, 0, 0, \dots]$$

$$x_2^* = [x_1 - \delta \lambda_1, x_2 - \delta \lambda_2, \dots, x_k - \delta \lambda_k, 0, 0, \dots]$$

satisfy

Also since $x_i > 0$, therefore taking δ such that

$$0 < \delta < \min_i \left\{ \frac{x_i}{|x_i|} \right\}$$

we conclude that first k components of x_1^* and x_2^* are always positive. But the remaining components of x_1^* and x_2^* are zero. which follows that x_1^* and x_2^* are feasible solutions from x .

Now

$$x_1^* + x_2^* = 2 [x_1, x_2, \dots, x_k, 0, 0, \dots, 0]$$
$$\text{or, } \frac{1}{2} x_1^* + \frac{1}{2} x_2^* = [x_1, x_2, \dots, x_k, 0, 0, \dots, 0] = x$$
$$\text{or, } x = \lambda x_1^* + (1-\lambda) x_2^* \quad \text{where } \lambda = \frac{1}{2}$$

i.e. x can be expressed as a convex combination of two distinct feasible solutions x_1^* and x_2^* . But this is a contradiction as x is an extreme point. Hence the vectors $\alpha_1, \alpha_2, \dots, \alpha_k$ are L.I.

Further we know that at most m vectors of E^m can be L.I. So $\alpha_1, \alpha_2, \dots, \alpha_k$ cannot be more than m and hence the extreme point x will have at most m non zero variables i.e. at least $(n-m)$ variables will be zero.

Thus x is a B.F.S. Hence every extreme point of the convex set of feasible solutions is a B.F.S.