Q. Find the continuity & differentiability of the function

$f(x) = x \sin\frac{1}{x}, x \neq 0$

$f(0) = 0$

at $x = 0$

Here $\lim\limits_{h\to 0} f(0+h) = \lim\limits_{h\to 0} (0+h)\sin\frac{1}{(0+h)} = 0$, Since $\sin\frac{1}{h}$ lies between -1 & 1

$\lim\limits_{h\to 0} f(0-h) = \lim\limits_{h\to 0} (0-h)\sin\frac{1}{(0-h)} = \lim\limits_{h\to 0} h\sin\frac{1}{h} = 0$

$f(0) = 0$

Hence $f(x)$ is continuous at $x = 0$

Also, $\lim\limits_{h\to 0} \dfrac{f(0+h) - f(0)}{h} = \lim\limits_{h\to 0} \dfrac{(0+h)\sin\frac{1}{(0+h)} - 0}{h} = \lim\limits_{h\to 0} \sin\frac{1}{h}$, which does not exist

$\lim\limits_{h\to 0} \dfrac{f(0-h) - f(0)}{-h} = \lim\limits_{h\to 0} \dfrac{(0-h)\sin\frac{1}{(0-h)} - 0}{-h} = \lim\limits_{h\to 0} \left(-\sin\frac{1}{h}\right)$, which does not exist

Hence $f(x)$ is discontinuous at $x = 0$.

Q. Find the continuity & discontinuity of the function

$f(x) = x$ when $0 < x < \frac{1}{2}$

$\quad 1$ when $x = \frac{1}{2}$

$\quad 1 - x$ when $\frac{1}{2} < x < 1$

at $x = \frac{1}{2}$

Here $\lim\limits_{h\to 0} f(\frac{1}{2}+h) = \lim\limits_{h\to 0} 1 - (\frac{1}{2}+h) = \frac{1}{2}$

$\lim\limits_{h\to 0} f(\frac{1}{2}-h) = \lim\limits_{h\to 0} (\frac{1}{2}-h) = \frac{1}{2}$

$f(\frac{1}{2}) = 1$.

Hence $f(x)$ is not continuous at $x = \frac{1}{2}$ & consequently $f(x)$ is not differentiable at $x = \frac{1}{2}$

Q. Examine the continuity & differentiability of the function

$$f(x) = 3+2x, \quad -\frac{3}{2} < x \le 0$$
$$3-2x, \quad 0 < x < \frac{3}{2}$$

at $x = 0$

Here $\lim_{h\to 0} f(0+h) = \lim_{h\to 0} 3-2(0+h) = 3$

$\lim_{h\to 0} f(0-h) = \lim_{h\to 0} 3+2(0-h) = 3$

& $f(0) = 3+2.0 = 3$

Hence $f(x)$ is continuous at $x=0$

Also, $$\lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{(3-2h)-3}{h} = -2$$

$$\lim_{h\to 0} \frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0} \frac{(3-2h)-3}{-h} = 2$$

Thus $$\lim_{h\to 0} \frac{f(0+h)-f(0)}{h} \neq \lim_{h\to 0} \frac{f(0-h)-f(0)}{-h}$$

& hence $f(x)$ is not differentiable at $x=0$

Q. Show that the function $f(x) = \dfrac{x}{1+e^{\frac{1}{x}}}$, $x \neq 0$ & $f(0)=0$ is not differentiable at $x=0$.

$$\lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{\frac{0+h}{1+e^{\frac{1}{0+h}}} - 0}{h} = \lim_{h\to 0} \frac{1}{1+e^{\frac{1}{h}}} = \frac{1}{1+e^{\infty}} = \frac{1}{\infty} = 0$$

$$\lim_{h\to 0} \frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0} \frac{\frac{0-h}{1+e^{\frac{1}{0-h}}} - 0}{-h} = \lim_{h\to 0} \frac{1}{1+e^{-\frac{1}{h}}} = \lim_{h\to 0} \frac{1}{1+\frac{1}{e^{\frac{1}{h}}}}$$

$$= \frac{1}{1+\frac{1}{e^{\infty}}} = \frac{1}{1+\frac{1}{\infty}} = \frac{1}{1+0} = 1$$

Hence $f(x)$ is not differentiable at $x=0$