Q. Prove that a function which is continuous in a closed interval $[a, b]$ is bounded therein

We know that if a function $f(x)$ is continuous in the closed interval $[a, b]$, then for a given $\epsilon > 0$ the interval can always be divided into a finite number of subintervals such that

$$|f(x_1) - f(x_2)| < \epsilon \quad \text{——— (1)}$$

where x_1 & x_2 are any two points in some subinterval.

Let the dividing points be

$$a = x_0, x_1, x_2, \ldots\ldots, x_{n-1}, x_n = b$$

Let x be any point in the first subinterval $[a, x_1]$

Then according to (1), we have

$$|f(a) - f(x)| < \epsilon \quad \text{——— (2)}$$

Now
$$\begin{aligned} |f(x)| &= |f(a) + \{f(x) - f(a)| \\ &\leq |f(a)| + |f(x) - f(a)| \\ &< |f(a)| + \epsilon \quad [\text{from } (2)] \end{aligned}$$

In particular when $x = x_1$, we have

$$|f(x_1)| < |f(a)| + \epsilon \quad \text{——— (3)}$$

Again let x be any point in the second subinterval $[x_1, x_2]$

Then according to (1), we have

$$|f(x_1) - f(x)| < \epsilon \quad \text{——— (4)}$$

Now
$$\begin{aligned} |f(x)| &= |f(x_1) + \{f(x) - f(x_1)\}| \\ &\leq |f(x_1)| + |f(x) - f(x_1)| \\ &< |f(x_1)| + \epsilon \quad [\text{from } (4)] \\ &< \{|f(a)| + \epsilon\} + \epsilon \quad [\text{from } (3)] \end{aligned}$$

i.e. $|f(x)| < |f(a)| + 2\epsilon$

In particular when $x = x_2$, we have

$$|f(x_2)| < |f(a)| + 2\epsilon$$

Proceeding in this way we find that when x is any point in the nth subinterval $[x_{n-1}, b]$, we have

$$|f(x)| < |f(a)| + n\epsilon$$

This inequality is true for the whole interval $[a, b]$ i.e. all the values of $f(x)$ in ~~the~~ interval $[a, b]$ lie between

$f(a) - n\epsilon$ & $f(a) + n\epsilon$

Hence $f(x)$ is bounded in $[a, b]$

Q. Disuss the continuity of the following function at $x = 0$

$$f(x) = \frac{\sin x}{x} \quad \text{when } x \neq 0$$
$$= 2 \quad \text{when } x = 0$$

We have

$$\lim_{h \to 0} f(0+h) = \lim_{h \to 0} \frac{\sin(0+h)}{0+h} = \lim_{h \to 0} \frac{\sin h}{h} = 1$$

$$\lim_{h \to 0} f(0-h) = \lim_{h \to 0} \frac{\sin(0-h)}{0-h} = \lim_{h \to 0} \frac{-\sin h}{-h} = 1$$

& $f(0) = 2$

Hence $f(x)$ is not continuous at $x = 0$.