Continuity & Differentiability

Continuity of a function at a point:

A function $f(x)$ is said to be continuous at a point $x = a$ if $\lim_{h \to 0} f(a+h) = \lim_{h \to 0} f(a-h) = f(a)$

Cauchy's definition:

A function $f(x)$ is said to be continuous at $x = a$ if for any arbitrary small positive number ϵ there exists a positive number δ such that $|f(x) - f(a)| < \epsilon$ for all values of x whenever $|x - a| < \delta$.

A function which is not continuous is called discontinuous function.

Differentiability of a function at a point:

A function $f(x)$ is called differentiable at a point $x = a$ if $\lim_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$ & $\lim_{h \to 0} \dfrac{f(a-h) - f(a)}{-h}$ both exist & have the same definite value.

The first limit is called right derivative of $f(x)$ at $x = a$ and is denoted by $Rf'(a)$ whereas the second limit is called left derivative of $f(x)$ at $x = a$ and is denoted by $Lf'(a)$.

Q. Prove that a function differentiable at a point must be continuous at that point but the converse is not necessarily true.

or

Prove that continuity of a function is a weaker condition than differentiability.

Let a function $f(x)$ be differentiable at a point $x = a$.

Then $$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = Rf'(a) \quad \text{--- (1)}$$

& $$\lim_{h \to 0} \frac{f(a-h) - f(a)}{-h} = Lf'(a) \quad \text{--- (2)}$$

both exist & have the same definite value.

We have

$$f(a+h)-f(a) = \frac{f(a+h)-f(a)}{h}\, h$$

$$\therefore \lim_{h\to 0}[f(a+h)-f(a)] = \lim_{h\to 0}\left[\frac{f(a+h)-f(a)}{h}\, h\right]$$

$$= \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} \lim_{h\to 0} h$$

$$= Rf'(a)\times 0 = 0 \quad [\text{from (1)}]$$

or, $\lim_{h\to 0} f(a+h) = f(a)$ —— (3)

Again, $f(a-h)-f(a) = \frac{f(a-h)-f(a)}{-h}(-h)$

$$\therefore \lim_{h\to 0}[f(a-h)-f(a)] = \lim_{h\to 0}\left[\frac{f(a-h)-f(a)}{-h}(-h)\right]$$

$$= \lim_{h\to 0}\frac{f(a-h)-f(a)}{-h} \lim_{h\to 0}(-h)$$

$$= Lf'(a)\times 0 = 0 \quad [\text{from (2)}]$$

or, $\lim_{h\to 0} f(a-h) = f(a)$ —— (4)

From (3) & (4) we have $\lim_{h\to 0} f(a+h) = \lim_{h\to 0} f(a-h) = f(a)$

Hence $f(x)$ is continuous at $x=a$.

For the converse we consider a function $f(x)=|x|$

We examine the continuity & differentiability of this function at $x=0$

We have $\lim_{h\to 0} f(0+h) = \lim_{h\to 0}|0+h| = 0$, $\lim_{h\to 0} f(0-h) = \lim_{h\to 0}|0-h| = 0$

& $f(0)=0$ which shows that $f(x)$ is continuous at $x=0$.

Now, $\lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{|0+h|-0}{h} = 1$

& $\lim_{h\to 0}\frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0}\frac{|0-h|-0}{-h} = -1$

which shows that $f(x)$ is not differentiable at $x=0$

Thus the converse of the theorem is not necessarily true.