Constraints

The limitations on the motion of a system are called constraints .The motion is said to be constrained motion . Constraints are always related to forces which restrict the motion of the system.These forces are called forces of constraint.

# Holonomic and Nonholonomic Constraints

# The nomenclature ‘holonomic’ constraints comes from the word ‘holos’which means ‘integer’ in Greek and ‘whole ‘ or intergrable’ in Latin languages .

# A system is said to be non-holonomic .If it corresponds to non-integrable differential equations of constrains .

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# Degrees of freedom and constraints

### Consider a system S with N particles, *Pr* (*r*=1,...,*N*), and their positions vector **x***r* in some reference frame A. The 3*N* components specify the configuration of the system, *S*.

The configuration space is defined as:

*T*

 **X**

**X**  *R*3*N* , **X**  **x**

.**a** , **x** .**a**

, **x** .**a**

,**x**

.**a** , **x**

.**a** , **x**

.**a**  

### The 3*N* scalar numbers are called configuration space variables or coordinates for the system.

1

1

1

2

1

3

*N*

1

*N*

2

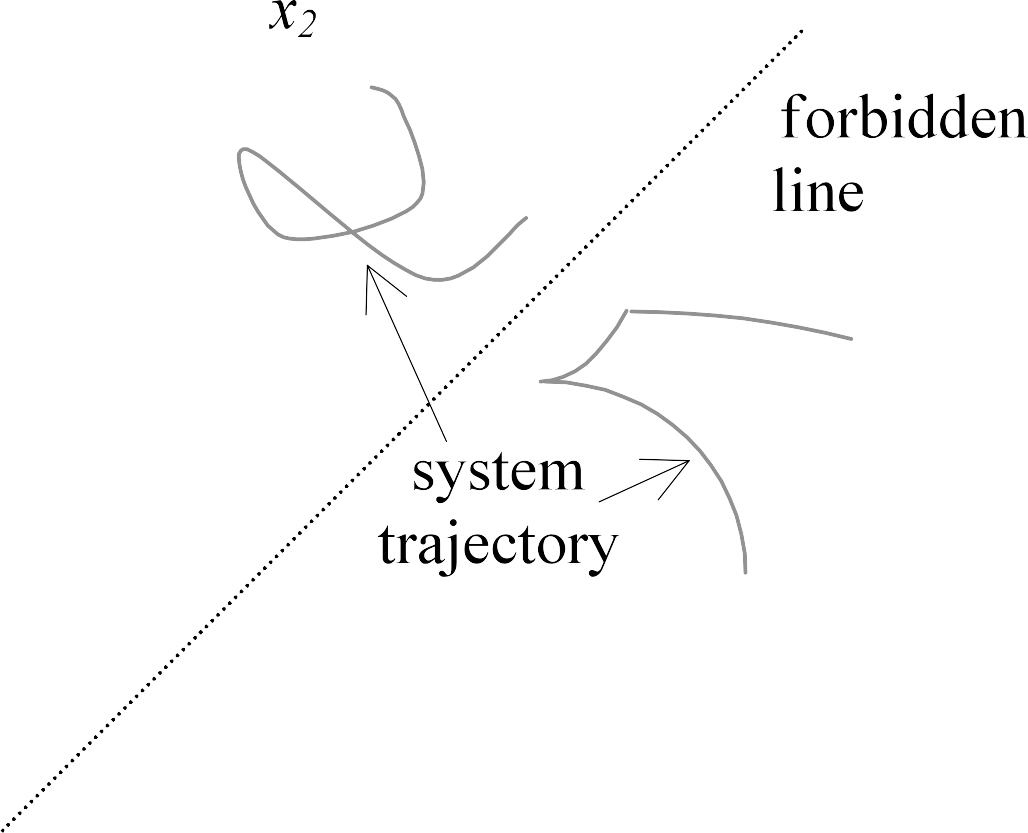
*N*

3

The trajectories of the system in the configuration space are always continuous.

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# A System of Two Particles on a Line



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# Holonomic Constraints

Constraints on the position (configuration) of a system of particles are called *holonomic* constraints.

* Constraints in which time explicitly enters into the constraint equation are called *rheonomic*.
* Constraints in which time is not explicitly present are called *scleronomic*.
* Particle is constrained to lie on a plane:

*A x*1 + *B x*2 + *C x*3 + *D* = 0

* A particle suspended from a taut string in three dimensional space.

(*x*1 – *a*)2 +(*x*2 – *b*)2 +(*x*3 – *c*)2 – *r*2 = 0

* A particle on spinning platter (carousel)

*x*1 = *a* cos(*t* + ); *x*2 = *a* sin(*t* + )

* + A particle constrained to move on a sphere in three-dimensional space whose radius changes with time *t*.

*x*1 *dx*1 + *x*2 *dx*2 + *x*3 *dx*3 - *c*2 *dt* = 0

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# Holonomic Constraint

*x*3

Configuration space

*f*(*x*1, *x*2, *x*3)=0

*x*2

*x*1

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**Scleronomic, holonomic**

**Rheonomic, holonomic**

#### t

*f*(*x*1, *x*2)=0 *f*(*x*1, *x*2, *t*)=0

*f*(*x*1, *x*2 , *t*4)=0

*f*(*x*1, *x*2 , *t*3)=0

Configuration space

*x*2

*f*(*x*1, *x*2)=0

*x*1 *x*1

*f*(*x*

*f*(*x*1, *x*2

, *x* , *t*

, *t*2)=0

*x*2

)=0

1 2 1

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Definition 1

# Nonholonomic Constraints

* A particle constrained to move on a

x

* All constraints that are not holonomic

Definition 2

* Constraints that constrain the velocities of particles but not their positions

We will use the second definition.

circle in three-dimensional space

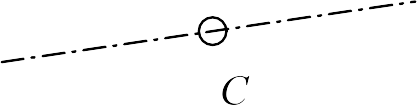
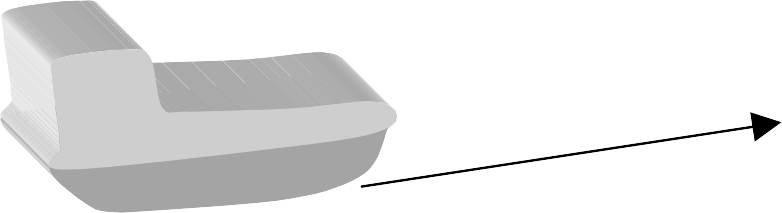
whose radius changes with time *t*. *x*1 *dx*1 + *x*2 *dx*2 + *x*3 *dx*3 - *c*2 *dt* = 0

* + The *knife-edge constraint*

****



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Aside: Inequality Constraints

### Holonomic or non holonomic?

* Inequalities do not constrain the position in the same way as equality constraints do.
* Rosenberg classifies inequalities as nonholonomic constraints.
* We will classify equality constraints into holonomic equality constraints and non holonomic equality constraints and treat inequality constraints separately

#### Inequalities in mechanics lead to complementarity constraints!

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# Complementarity Constraints

separation, 

normal force, 



**  0, **  0

**  0, **  0

**  0

More generally,

0  **  **  0

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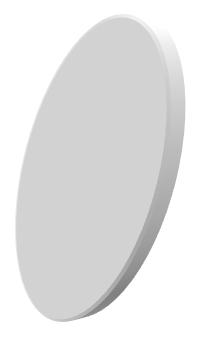
# Examples of Velocity Constraints

### Example 1

#### Are these configuration constraints?

A particle moving in a horizontal plane (call it the x-y plane) is steered in such a way that the slope of the trajectory is proportional to the time elapsed from t=0.

*z*



Example 2: Disk rolling on plane

**a**1

**b**

**e**2

3

*C*

*C\**

*q4*

*q1*

*q3*

**b**1

*P*

**e**3

**e**1

Rolling constraint at *P*

**a**3

*q2*

**a**2

**b**2

*y*

*x*

*q5*

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# When is a constraint on the motion nonholonomic?

Velocity constraint



We should be familiar with this question in the 3 dimensional case

Or constraint on instantaneous motion



*P dx* + *Q dy* + *Rdz* =0

**v**





### Pfaffian Form

Question

Can the above equation can be reduced to the form:

*f*(*x*1, *x*2, ..., *xn*-1, *t*) = 0

Can we construct a surface in 3-D whose normal at every point is given by **v**?

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### When is a scleronomic constraint on motion in a three-dimensional configuration space nonholonomic?

Velocity constraint



Or constraint in the Pfaffian form

*P dx* + *Q dy* + *Rdz* =0 (1)

Question

Can the above equation can be reduced to the form:

*f*(*x*, *y*, *z*)=0

Or,

Can we at least say when the differential form (1) an exact differential?

*df* = *P dx* + *Q dy* + *Rdz*

* A **sufficient** condition for (1) to be integrable is that the differential form is an exact differential.
* If it is an exact differential, there must exist a function *f*, such that
* The necessary and sufficient conditions for this to be true is that the first partial derivatives of *P*, *Q*, and *R* with respect to *x*, *y*, and *z* exist, and



Recall result from Stokes Theorem!

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Exactness and independent of path

### If **v** is continuous and has continuous first partials in a domain *D, and* the line integral



is independent of path *C* in *D* (that is, **v.***d***r** is *exact*) then

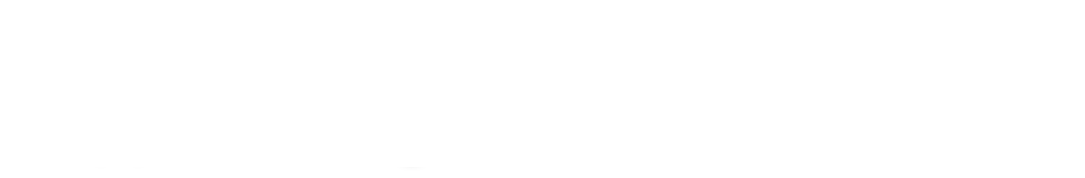
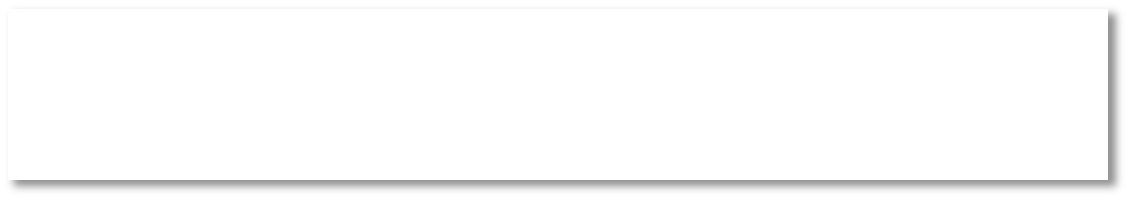
### (2)



*C’’*

*C’*

*C*



But (2) is only a **sufficient** condition for

(1) to be integrable.

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### Necessary and sufficient condition for a motion constraint in 3-D space to be holonomic

Can the constraint in the Pfaffian form

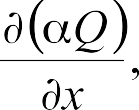
*P dx* + *Q dy* + *Rdz* =0 (1) be reduced to the form:

*f*(*x*, *y*, *z*)=0

For the constraint to be *integrable*, it is necessary and sufficient that there exist an integrating factor (*x*, *y*, *z*), such that,

* If (3) is an exact differential, there must exist a function *g*, such that
  + The necessary and sufficient conditions for this to be true is that the first partial derivatives of *P*, *Q*, and *R* with respect to *x*, *y*, and *z* exist, and



*P dx* + *Q dy* + *Rdz* =0 (3) be an exact differential.



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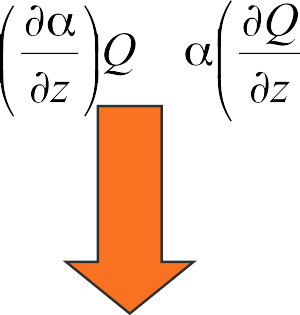
### Does there exist  such that:







### Necessary and sufficient condition for



(2) to be holonomic, provided **v** is a well-behaved vector field and



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1. sin *x*3 *dx*1 - cos *x*3 *dx*2 = 0

## Examples



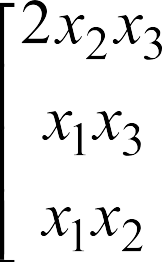


2. 2*x*2*x*3 *dx*1 + *x*1*x*3 *dx*2 + *x*1*x*2 *dx*3 = 0

*x*1 (2*x*2*x*3 *dx*1 + *x*1*x*3 *dx*2 + *x*1*x*2 *dx*3) = 0



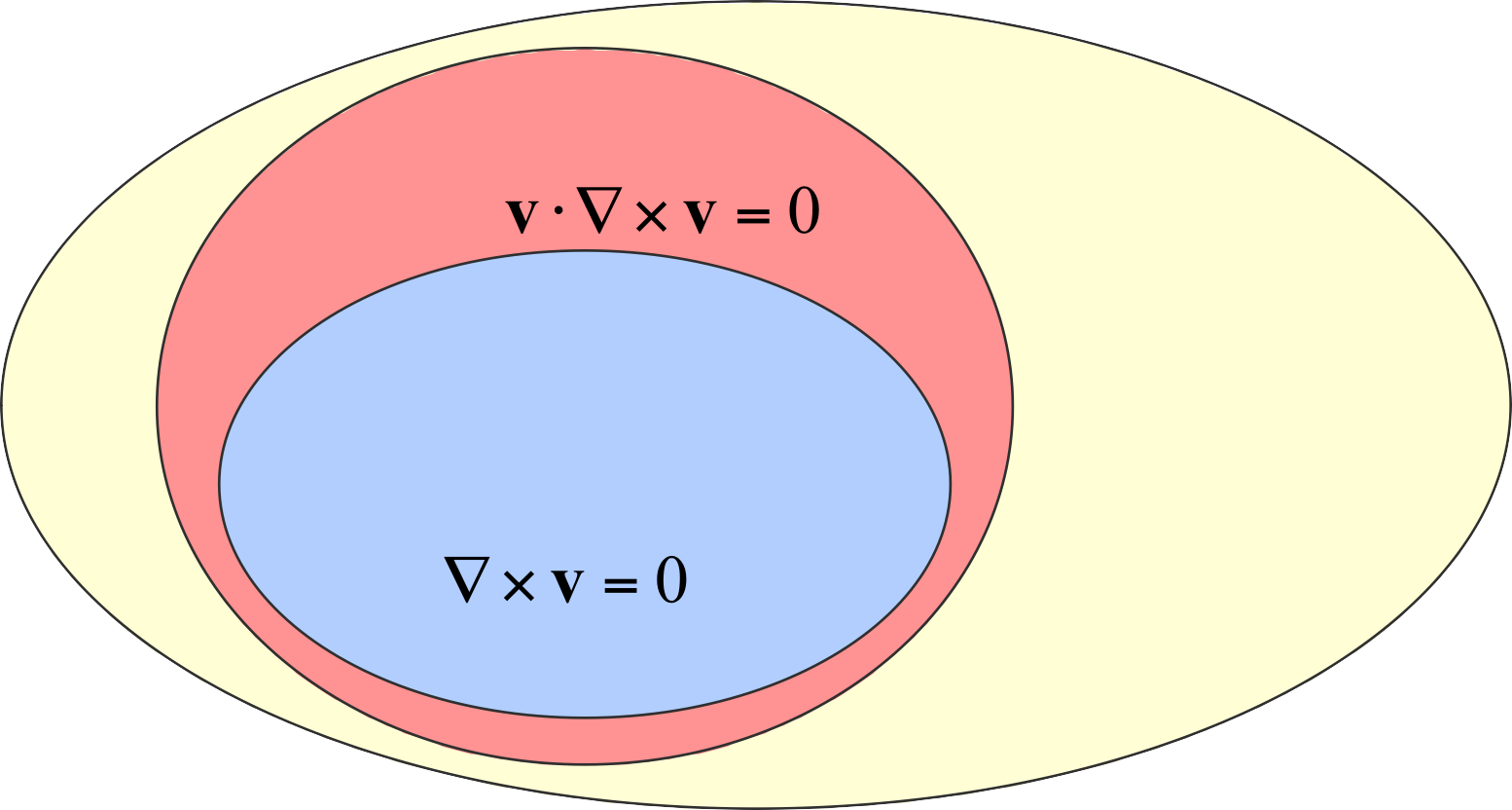
3.



*d*((*x*1)2 *x*2 *x*3) = 0

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# Nonholonomic constraints in 3-D



*Other nonholonomic constraints*

Holonomic

Nonholonomic

Holonomic

*P dx* + *Q dy* + *Rdz* =0

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# Extension to 2-D rheonomic constraints

### Compare

Can the constraint of the form

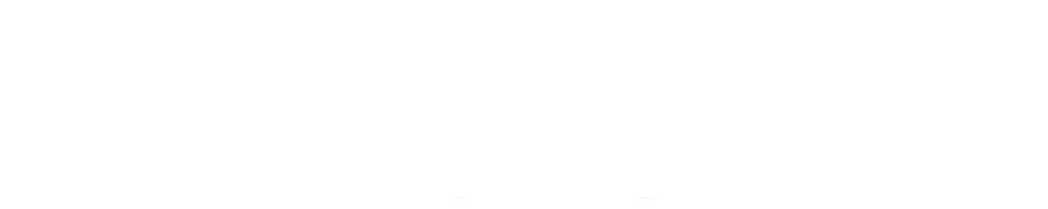
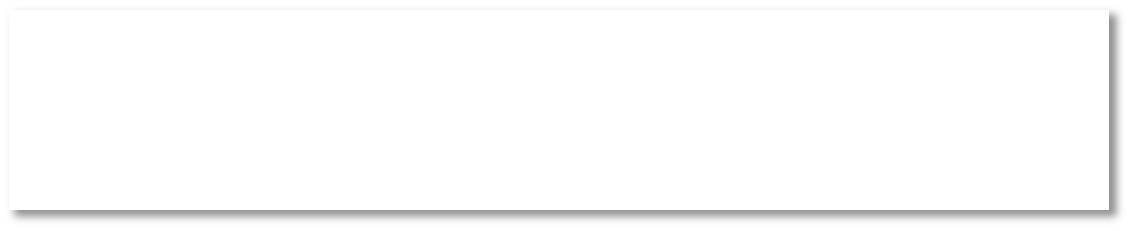
*P dx* + *Q dy* + *Rdz* =0 be reduced to the form:

*f*(*x*, *y*, *z*)=0

### Can the constraint of the form

*P dx* + *Q dy* + *Rdt* =0 be reduced to the form:

*f*(*x*, *y*, *t*)=0



Necessary and sufficient condition is the same if you replace *z* with *t*

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d*x2* – *x*3 d*x*1 = 0 and

d*x3* – *x*1 d*x*2 = 0

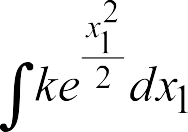
Multiple Constraints

### Are the constraint equations non holonomic?

Individually: YES!

### Together:

d*x3* – *x*1 d*x*2 = d*x3* – *x*1 (*x*3 d*x*1) = 0



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## Frobenius Theorem: Generalization to *n* dimensions

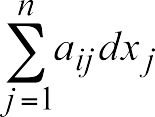
*n* dimensional configuration space

**v**

**w**

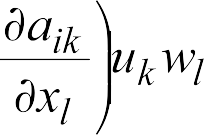
**u**

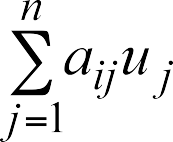
*m* independent constraints (*i*=1,..., *m*)



The necessary and sufficient condition for the existence of *m* independent equations of the form:

*fi*(*x*1, *x*2, ..., *xn*) = 0, *i*=1,..., *m.*

is that the following equations be satisfied:





where *uk* and *wl* are components of any two *n* vectors that lie in the null space of the *m*x*n* coefficient matrix **A** = [*aij*]:



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### Generalized Coordinates and Number of Degrees of Freedom

**Number of degrees of freedom of a holonomic system** in any reference frame

* the minimum number of variables to completely specify the position of every particle in the system in the chosen reference

The variables are called generalized coordinates

No. of degrees of freedom

= No. of variables required to describe the system

- No. of independent configuration constraints

There can be no holonomic constraint equations that constrain\* the values the generalized coordinates can have.

*q*1, *q*2, ..., *qn* denote the generalized coordinates for a system with *n* degrees of freedom in a reference frame *A*.

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## Degrees of Freedom: Example



*A*

*P*

*Q*



*A*

*P*

*Q*



*A*

*P*

*Q*

No. of degrees of freedom = No. of variables required to describe the system

- No. of independent configuration constraints

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# Generalized Coordinates and Speeds

**Holonomic Systems**

Number of degrees of freedom of a system in any reference frame

* the minimum number of variables to completely specify the position of every particle in the system in the chosen reference

The variables are called generalized coordinates

*q*1, *q*2, ..., *qn* denote the generalized coordinates for a system with *n* degrees of freedom in a reference frame *A*.

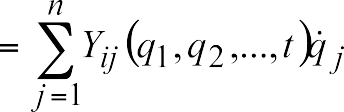
*n* generalized coordinates specify the position (configuration of the system)

For a holonomic system, the number of independent speeds describing the rate of change of configuration of the system is also equal to *n*

In a system with *n* degrees of freedom in a reference frame *A*, there are n scalar quantities, *u*1, *u*2, ..., *un* (for that reference

frame) called generalized speeds. They that

are related to the derivatives of the generalized coordinates by :



where the *nxn* matrix **Y** = [*Yij*] is non singular and **Z** is a *nx*1 vector.

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Generalized Coordinates

*q*1 , *q*2 , *q*3 , *q*4 , *q*5

Generalized Speeds

# Example 1

*z*

*A******C* = *u1* **b***1* + *u2* **b***2* + *u3* **b***3*

*u4* = derivative of *q4 u5* = derivative of *q5*

*x q1*

*q5*

*q2*

**b**2

**b**3 *C*

*C\**

*q3* **b**1

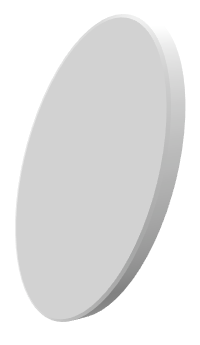
*P*

*y*

*q4*

Locus of the

point of contact *Q* on the plane *A*



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# Example 2

Bug moving radially on a turntable that can rotate 0)

Generalized coordinates in *A*

* *s*, 
* *x*1, *x*2

Generalized speeds



 or





*Q*

**b** *s*

2



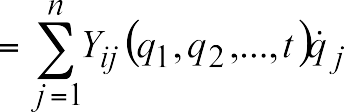
**b**1



*O*



Generalized speeds and derivatives of generalized coordinates

**a**2 





**a**1 Appears in

rheonomic constraints

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# Example 2: Turntable angular velocity is given

Bug moving radially on a rotating turntable 0)

Generalized coordinates in *A*

* *s*
* *x*1

Generalized speeds

or





Generalized speeds and derivatives of generalized coordinates





*Q*

**b** *s*

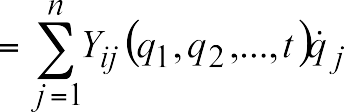
2



**b**1



*O*

**a**2 





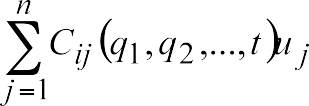
**a**1 Appears in

rheonomic constraints

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### Nonholonomic Constraints are Written in Terms of Speeds

*m* constraints in *n* speeds

*m* speeds are written in terms of the *n*-*m*

1. independent speeds

Define the *number of degrees of freedom for a nonholonomic system* in a reference frame *A* as *p*, the number of independent speeds that are required to completely specify the velocity of any particle belonging to the system, in the reference frame *A*.

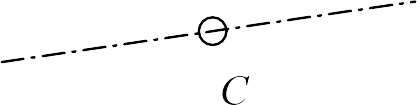
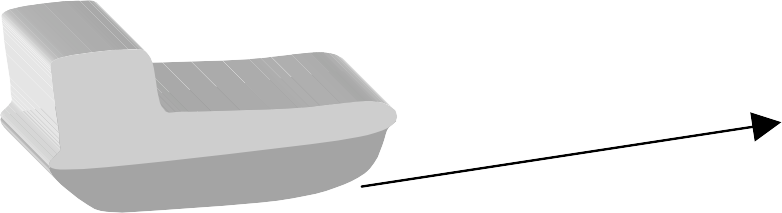


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Number of degrees of freedom

# Example 3

* + *n* – *m =* 2 degrees of freedom



Generalized coordinates

* + (*x*1, *x*2, *x*3)

Speeds: Choice 1

* + forward velocity along the axis of the skate, *vf*
  + the speed of rotation about the vertical axis, 



* + and the lateral (skid) velocity in the transverse direction, *vl*



Speeds: Choice 2

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