Exp 17/04/2021 Linear Algebra

In R^4 the set $\{(1,0,1,0), (1,1,1,1)$ and $(0,1,3,1)\}$ be basis of vector space V. Then process to compute (construct) the orthogonal vectors w_1, w_2, w_3 and then we normalize these constructed vector to obtain an orthonormal set.

Solution We take $w_1 = v_1 = (1,0,1,0)$

$v_1 = (1,0,1,0)$

$v_2 = (1,1,1,1)$

$v_3 = (0,1,3,1)$

Then construct/compute orthogonal vector by using Gram-Schmidt process.

$$w_k = v_k - \sum_{i=1}^{k-1} \frac{\langle v_k, w_i \rangle}{\|w_i\|^2} \cdot w_i$$

$w_1 = (1,0,1,0)$

$$w_2 = v_2 - \frac{\langle v_2, w_1 \rangle}{\|w_1\|^2} \cdot w_1$$

$$= (1,1,1,1) - \frac{\langle (1,1,1,1), (1,0,1,0) \rangle}{\langle (1,0,1,0), (1,0,1,0) \rangle} (1,0,1,0)$$

$$= (1,1,1,1) - \frac{(1+0+1+0)}{(1+0+1+0)} (1,0,1,0)$$

$$= (1,1,1,1) - \frac{2}{2} (1,0,1,0)$$

$$= (1,1,1,1) - (1,0,1,0)$$

$w_2 = (0,1,0,1)$

Now compute w_3

$$w_3 = v_3 - \frac{\langle v_3, w_1\rangle}{\|w_1\|^2} w_1 - \frac{\langle v_3, w_2\rangle}{\|w_2\|^2} w_2$$

$\langle v_3, w_1\rangle = \langle (0,1,2,1), (1,0,1,0)\rangle$

$\langle v_3, w_1\rangle = 0+0+2+0 = 2$

$\langle v_3, w_2\rangle = \langle (0,1,2,1), (0,1,0,1)\rangle$

$\langle v_3, w_2\rangle = 0+1+0+1 = 2$

$\|w_1\|^2 = 1^2+0+1^2+0 = 2 \quad \Rightarrow \|w_1\| = \sqrt{2} \quad \leftarrow (1)$

$\|w_2\|^2 = 0+1^2+0+1^2 = 2 \quad \Rightarrow \|w_2\| = \sqrt{2} \quad \leftarrow (2)$

$w_3 = (0,1,2,1) - \frac{2}{2}(1,0,1,0) - \frac{2}{2}(0,1,0,1)$

$= (0,1,2,1) - [(1,0,1,0) + (0,1,0,1)]$

$= (0,1,2,1) - (1+0, 0+1, 1+0, 0+1)$

$= (0,1,2,1) - (1,1,1,1)$

$w_3 = (-1, 0, 1, 0) \quad \Rightarrow \|w_3\|^2 = 1+0+1+0 = 2$

$\|w_3\| = \sqrt{2}$

The set of vectors w_1, w_2, w_3 ar orthogonal mutually then $\langle w_i, w_j\rangle = 0$

$\langle w_1, w_2\rangle = \langle (1,0,1,0), (0,1,0,1)\rangle = 0+0+0+0 = 0$

$\langle w_2, w_3\rangle = \langle (0,1,0,1), (-1,0,1,0)\rangle = 0+0+0+0 = 0$

$\langle w_3, w_1\rangle = \langle (-1,0,1,0), (1,0,1,0)\rangle = -1+0+1+0 = 0$

Hence w_1, w_2, w_3 are orthogonal set of vectors.

These vector can be normalized then obtain orthonormal set of vectors.

$$u_1 = \frac{w_1}{\|w_1\|} = \frac{1}{\sqrt{2}}(1,0,1,0) = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0\right)$$

$$u_2 = \frac{w_2}{\|w_2\|} = \frac{1}{\sqrt{2}}(0,1,0,1) = \left(0, \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$$

$$u_3 = \frac{w_3}{\|w_3\|} = \frac{1}{\sqrt{2}}(-1,0,1,0) = \left(-\frac{1}{\sqrt{2}}, 0\ \frac{1}{\sqrt{2}}, 0\right)$$

Hence u_1, u_2, u_3 set of vectors are orthonormal.