Conjugate class :- since an equivalence relation defined on a set decompose the set mutually disjoint equivalence classes, hence the relation of conjugacy, defined on a group G decompose G into mutually disjoint equivalence classes known as classes of conjugate element.

Let C_a denote the equivalence class determined by an element $a \in G$, then

$$C_a = \{x \in G : x = a\} = \{x \in G : x = y^{-1}ay \text{ for some } y \in G\}$$
$$= \{y^{-1}ay : y \in G\}$$

Here, C_a is defined as conjugate class of a in G. Also C_a is the class element conjugate to a.

Theorem :- Let G be a finite group and $a \in G$. Let $N(a)$ denotes the normalizer of a in G and C_a the conjugate class of a in G. Then $O(C_a) = \dfrac{O(G)}{O(N(a))}$ i.e. number of distinct elements conjugate to a in G is the index of the normalizer of a in G.

Proof :- Let G be a finite group and $a \in G$.

To show $O(C_a) = \dfrac{O(G)}{O(N(a))}$

By definition, we have

$$N(a) = \{x \in G : ax = xa\}$$

and $C_a = \{x^{-1}ax : x \in G\}$

We write $M = \{N(a)x : x \in G\}$

= Set of right Cosets of $N(a)$ in G.

Define a map $f : M \to C_a$ such that $f\{N(a) : x\} = x^{-1}ax \ \forall x \in G$, obviously f is well defined.

To show f is one-one and onto.

f is one-one

Consider $f[N(a).x] = f[N(a).y] : x, y \in G$

$\Rightarrow x^{-1}ax = y^{-1}ay$

$\Rightarrow x(x^{-1}ax)y^{-1} = x(y^{-1}ay)y^{-1}$

$\Rightarrow (xx^{-1})(axy^{-1}) = (xy^{-1}a)e$

$\Rightarrow axy^{-1} = xy^{-1}a \Rightarrow a(xy^{-1}) = (xy^{-1})a$

$\Rightarrow xy^{-1} \in N(a) \Rightarrow N(a)xy^{-1} = N(a)$

$\Rightarrow N(a).x = N(a).y$

$\Rightarrow f$ is one-one

f is onto.

For given any $x^{-1}ax \in C_a \ \exists \ N(a).x \in M$ such that $f[N(a).x] = x^{-1}ax$

$\Rightarrow f$ is onto.

Hence $o(C_a) = o(M)$ = number of distinct right Cosets of $N(a)$ in G.

= index of $N(a)$ in G.

$= \dfrac{o(G)}{o\,N(a)}$, by definition of index.

Conjugate class :- It is explained in Normalizer of an element notes.

class equation

Theorem :- If G is a finite group, then $O(G) = \sum \frac{O(G)}{O N(a)}$ where this sum is taken over one element of each Conjugate class.

Proof :- since, we know that, the relation of Conjugacy is an equivalence relation on the set G.

$\Rightarrow$ This relation partitions G into disjoint equivalence classes.

Let C_a, C_b, C_c respectively denote Conjugate classes of elements $a, b, c \in G$. Also let

$$O(C_a) = c_a, \; O(C_b) = c_b, \; O(C_c) = c_c$$

Then $G = C_a \cup C_b \cup C_c \cdots$

$$\Rightarrow O(G) = O(C_a) + O(C_b) + O(C_c) + \cdots$$

($\because C_a, C_b, C_c$ are mutually disjoint)

where the sum runs over each elements of Conjugate class.

But $c_a = \frac{O(G)}{O(N(a))}$

$$\therefore O(G) = \sum \frac{O(G)}{O N(a)} \quad \text{——} \; (i)$$

The above equation (i) is known as class equation of G.