Characteristic of a field

Let F be a field. A least positive integer n is called the characteristic of F if any non-zero element (particularly the unity element) of F is of order n i.e. $na = 0 \; \forall \; a \in F$, where 0 is the additive identity of F.

If no such positive integer is available, we say that F has characteristic zero.

Q. Prove that the characteristic of a field is either zero or a prime number according as the unity element e regarded as a member of the additive group of the field is of order zero or a prime number

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Let F be a field & e be its unity element

If $O(e) = 0$, then the characteristic of F is zero.

Now let $O(e) = p$, where p is the least positive integer such that $pe = 0$

Let a be any element of F

Then $pa = p(ea)$

$= (pe)a$

$= 0a$

$= 0$

Thus p is the least positive integer such that $pa = 0$

So by definition, the characteristic of F is p.

We shall now prove that p is a prime number.

If possible, let us suppose that p is not prime.

Then p is composite integer. So we can write

$p = p_1 p_2$, where $p_1 > 1$ & $p_2 < p$

The characteristic of F is p

$\Rightarrow o(e) = p$

$\Rightarrow pe = 0$

$\Rightarrow p_1 p_2 e = 0$

$\Rightarrow p_1 (p_2 e) = 0$

$\Rightarrow (p_1 e)(p_2 e) = 0$

$\Rightarrow p_1 e = 0$ or $p_2 e = 0$

$\Rightarrow$ Characteristic of F is either p_1 or $p_2 < p$

$\Rightarrow$ Characteristic of $F < p$

Thus we have

Characteristic of F is $p \Rightarrow$ Characteristic of $F < p$, which is a contradiction.

Hence our supposition that p is not prime is wrong.

Therefore p is prime.

Thus the characteristic of a field is either zero or a prime number.

Q. If x, y are any elements of a field with characteristic $n \neq 0$, then show that $x^n + y^n = (x+y)^n$

Let F be a field & e its unity element. Since n is the characteristic of F, then $ne = 0$ —— (1)

Let $x, y \in F$. Then $x^n, y^n \in F \ \forall n \in N$

From Binomial theorem we know that

$$(x+y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{\lfloor 2} x^{n-2}y^2 + \cdots\cdots + y^n$$

Using (1) we get

$$(x+y)^n = x^n + 0 + 0 + \cdots\cdots + y^n$$

or, $(x+y)^n = x^n + y^n$